Schneider-Stuhler complex of locally analytic principal series for PGL_n.

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Background

Let $E/\mathbb{Q}_p$ be a finite field extension, $G := \mathrm{GL}_n(\mathbb{Q}_p)$, and X be its Bruhat-Tits building: a G-simplicial complex whose vertices are homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ and whose q-simplices are represented by descending chains of lattices in $\mathbb{Q}_p^n$ of length q

$$\Lambda_0 \supsetneq \Lambda_1 \supsetneq \ldots \Lambda_q \supsetneq p\Lambda_0$$

For each simplex F of X, we have a filtration of its stabilizer $\mathcal{P}_F$ by subgroups $\mathcal{P}_F \supsetneq U_F^{(0)} \supsetneq U_F^{(1)} \cdots \supsetneq U_F^{(e)} \supsetneq \ldots$ with nice properties such as $U_F^{(e)} \subset U_{F'}^{(e)}$ if $F \subset F'$. So, if V is any complex smooth representation of G, then $V^{U_{F'}^{(e)}} \subset V^{U_F^{(e)}}$, and we get a chain complex of oriented G-chains valued in various U_F-fixed subspaces of V, for simplices F in X.
We call it the (smooth) Schneider-Stuhler complex for V at level e.

Problem

Theorem (Schneider-Stuhler, 1993)

Let $U := U^{(e)}_{[\mathbb{Z}_p^n]} = \left\{ \mathrm{GL}_n(\mathbb{Z}_p) \ni A \equiv I_n \pmod{p^e} \right\}$. For any V generated by V^U as G-representation, the Schneider-Stuhler complex $\mathrm{SS}_\bullet := C_c^{or}(X_\bullet, \underline{V})$ is a resolution of V.

We may consider the larger category of locally $\mathbb{Q}_p$-analytic E-linear G-representations, and replace the fixed vectors V^{U_F} by the analytic vectors $V^{U_F\text{-an}}$. Then we again get a chain complex.

Question

Suppose V is generated by $V^{U\text{-an}}$. Is it true that its locally analytic Schneider-Stuhler complex gives a resolution of V?

Related results for locally analytic principal series V

Theorem (Lahiri, 2020)

For $n = 2$, i.e., $G = \mathrm{GL}_2(\mathbb{Q}_p)$, let B be the upper-triangular Borel subgroup, T the diagonal torus, and $V = \mathrm{Ind}_B^G(\chi)$ the locally analytic induction of any locally $\mathbb{Q}_p$-analytic character of T. Then, there exists $e \in \mathbb{Z}_{\geq 0}$ depending on χ such that the locally analytic Schneider-Stuhler complex for V at level e is a resolution of V.

The proof is a detailed analysis of the action of G on $G/B = \mathbb{P}^1$.

Remark

The Schneider-Stuhler complex can be defined in exactly the same way for char p smooth V, but it is *not* always a resolution of V ($\exists$ counterexample by Ollivier-Schneider, using results of Breuil). However, for smooth principal series V, Ollivier (2013) has shown that it *is* a resolution regardless of characteristic.

Related results for locally analytic principal series V

Theorem (Kohlhaase-Schraen, 2013)

Let $I \subset G := \mathrm{PGL}_n(\mathbb{Q}_p)$ be the Iwahori subgroup, i.e., represented by those in $\mathrm{PGL}_n(\mathbb{Z}_p)$ which are upper-triangular modulo p. Let $U^{(e)}$ be the filtration of $\mathrm{PGL}_n(\mathbb{Z}_p)$. Let $\mathcal{A} := \mathrm{Ind}_{I\cap B}^{I}(\chi)^{U^{(e)}\text{-an}}$, which is naturally isomorphic to $\mathrm{Ind}_B^G(\chi)(I)^{U^{(e)}\text{-an}} = V(I)^{U^{(e)}\text{-an}}$. Then, $\exists$ an exact sequence of locally analytic G-representations

$$\left(\bigwedge\nolimits^{\bullet} E^{\Delta}\right) \otimes_E \mathrm{c\text{-}Ind}_I^G(\mathcal{A}) \to V = \mathrm{Ind}_B^G(\chi) \to 0$$

for large enough e, where Δ is the set of simple roots of G.

To each $\alpha \in \Delta$, the authors define $y_\alpha \in \mathrm{End}_G(\mathrm{c\text{-}Ind}_I^G(\mathcal{A}))$. Then, the resolution is the Koszul complex of $\mathrm{c\text{-}Ind}_I^G(\mathcal{A})$ defined by these endomorphisms.

Related results for locally analytic principal series V

Since the two complexes have the same length n and the same 0-th homology V, one may hope to relate them. Indeed, we can show

Proposition

For $n = 2$, there is a $\mathrm{PGL}_2(\mathbb{Q}_p)$-equivariant isomorphism from the Kohlhaase-Schraen resolution to the Schneider-Stuhler complex.

The strategy is to rewrite the Schneider-Stuhler complex in terms of compact inductions from the stabilizer of the stabilizer $\mathcal{P}_F$ of representatives F for G-orbits to the whole group G. Then, the natural maps turned out to be isomorphisms.

Remark

For $n \geq 3$, if there is a morphism of complexes $\mathrm{KS}_\bullet \to \mathrm{SS}_\bullet$, it seems that it is not an isomorphism degree-wise.

Consequences and the next step

Application

There is a work in progress by Shishir Agrawal and Matthias Strauch, which, assuming the exactness of the Schneider-Stuhler complex, uses the theory of solid locally analytic representations and a spectral sequence argument to show that if V and W are admissible, then the Schneider-Stuhler resolution of V can be used to compute $\mathrm{Ext}_G(V, W)$.

Future

The next step is to see if the arguments of Ollivier can be adapted to locally analytic principal series.