Schneider-Stuhler complex of locally analytic principal series for PGL_n.

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We call it the (smooth) Schneider-Stuhler complex for V at level e.

Problem

Theorem (Schneider-Stuhler, 1993)

Let $U := U^{(e)}_{[\mathbb{Z}_p^n]} = \left\{ \mathrm{GL}_n(\mathbb{Z}_p) \ni A \equiv I_n \pmod{p^e} \right\}$. For any V generated by V^U as G-representation, the Schneider-Stuhler complex $\mathrm{SS}_\bullet := C_c^{or}(X_\bullet, \underline{V})$ is a resolution of V.

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We may consider the larger category of locally $\mathbb{Q}_p$-analytic E-linear G-representations, and replace the fixed vectors V^{U_F} by the analytic vectors $V^{U_F\text{-an}}$. Then we again get a chain complex.

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Question

Suppose V is generated by $V^{U\text{-an}}$. Is it true that its locally analytic Schneider-Stuhler complex gives a resolution of V?

Related results for locally analytic principal series V

Theorem (Lahiri, 2020)

For $n = 2$, i.e., $G = \mathrm{GL}_2(\mathbb{Q}_p)$, let B be the upper-triangular Borel subgroup, T the diagonal torus, and $V = \mathrm{Ind}_B^G(\chi)$ the locally analytic induction of any locally $\mathbb{Q}_p$-analytic character of T.

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Let $I \subset G := \mathrm{PGL}_n(\mathbb{Q}_p)$ be the Iwahori subgroup, i.e., represented by those in $\mathrm{PGL}_n(\mathbb{Z}_p)$ which are upper-triangular modulo p.

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$$\left(\bigwedge\nolimits^{\bullet} E^{\Delta}\right) \otimes_E \text{c-}\mathrm{Ind}_I^G(\mathcal{A}) \to V = \mathrm{Ind}_B^G(\chi) \to 0$$

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To each $\alpha \in \Delta$, the authors define $y_\alpha \in \mathrm{End}_G(\mathrm{c\text{-}Ind}_I^G(\mathcal{A}))$. Then, the resolution is the Koszul complex of $\mathrm{c\text{-}Ind}_I^G(\mathcal{A})$ defined by these endomorphisms.

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Since the two complexes have the same length n and the same 0-th homology V, one may hope to relate them. Indeed, we can show

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For $n = 2$, there is a $\mathrm{PGL}_2(\mathbb{Q}_p)$-equivariant isomorphism from the Kohlhaase-Schraen resolution to the Schneider-Stuhler complex.

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The strategy is to rewrite the Schneider-Stuhler complex in terms of compact inductions from the stabilizer of the stabilizer $\mathcal{P}_F$ of representatives F for G-orbits to the whole group G. Then, the natural maps turned out to be isomorphisms.

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Remark

For $n \geq 3$, if there is a morphism of complexes $\mathrm{KS}_\bullet \to \mathrm{SS}_\bullet$, it seems that it is not an isomorphism degree-wise.

Consequences and the next step

Application

There is a work in progress by Shishir Agrawal and Matthias Strauch, which, assuming the exactness of the Schneider-Stuhler complex, uses the theory of solid locally analytic representations and a spectral sequence argument to show that if V and W are admissible, then the Schneider-Stuhler resolution of V can be used to compute $\mathrm{Ext}_G(V, W)$.

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Future

The next step is to see if the arguments of Ollivier can be adapted to locally analytic principal series.