CHANGE OF WEIGHTS OPERATIONS FOR TRIANGULATED (φ, Γ)-MODULES

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Abstract. Wu has shown the existence of "change of weights" operation on (φ, Γ)-modules in families, [Wu, Prop. 3.16]. We interpret it in the triangulline case as pullbacks with a discussion on related stacks. Finally, we prove that it intertwines well with translation functors via a 1-1 correspondence defined by Ding [Din25] in the non-critical crystabelline case.

Contents

1. Introduction

1.1. **Background and main results.** Under the locally analytic p-adic Langlands correspondence of $\mathrm{GL}_2(\mathbb{Q}_p)$ established in [Col16], the papers [JLS24] and more generally [Dina] studied certain "change of weights" operations on 2-dimensional p-adic $G_{\mathbb{Q}_p}$-representations (or rather, (φ, Γ)-modules of rank 2 over the Robba ring) that were shown to correspond to translation functors ([Hum08]) on the side of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then, such operations were generalized by Wu to families of (φ, Γ)-modules of rank 2 over rigid spaces in [Wu]. The paper [Col19] can be seen as a precursor of these techniques and results (even though the connection to translation functors is not considered there).

1.1.1. *Change of weights for triangulated (φ,Γ_K)-modules.* It is natural to ask for a general theory of "change of weights" for (φ,Γ_K)-modules. For $\mathrm{GL}_2(\mathbb{Q}_p)$, using the basic constructions in the p-adic Langlands correspondence of Colmez, Ding equipped $(\varphi,\Gamma_{\mathbb{Q}_p})$-modules D of rank 2 over the Robba ring $\mathcal{R}_{\mathbb{Q}_p,E}$ with $\mathfrak{gl}_2(\mathbb{Q}_p)$-module structures, equipped $D\otimes_E \mathrm{Sym}^k(E^2)$ with natural $\mathfrak{gl}_2(\mathbb{Q}_p)$-module and $(\varphi,\Gamma_{\mathbb{Q}_p})$-module structures, and developed a theory of change of weights in [Dina].

For $n>2$ or a larger base field $K\neq\mathbb{Q}_p$, it is unclear to us how to equip (φ,Γ_K)-modules over $\mathcal{R}_{K,E}$ with $\mathfrak{gl}_n(K)$-module structures, but we always have pullback and pushforward operations in the category of pairs $(D,\mathrm{Fil}^\bullet(D))$, where D is a trianguline (φ,Γ_K)-module and $\mathrm{Fil}^\bullet(D)$ is a triangulation on D, i.e., a filtration of D with successive quotients of rank 1. The bijectivity of their induced maps between the cohomology groups in [Col08, Theorem 2.22(i)] is reminiscent of the fact in [Hum08, §7.8] that translation functors between dominant integral weights of the same regularity are equivalences of categories, which motivated us to consider pullback and pushforward operations as some kind of "change of weights" operations.

More precisely, given a triangulated (φ,Γ_K)-module $(D,\mathrm{Fil}^\bullet(D))$ of rank n over $\mathcal{R}_{K,A}$ with

$$\mathrm{Fil}^0(D)=0\subsetneq \mathrm{Fil}^1(D)\subsetneq\cdots\subsetneq\mathrm{Fil}^n(D)=D,$$

where A is any affinoid algebra over a finite extension $E/\mathbb{Q}_p$ and $\mathrm{Fil}^i(D)$ are saturated (φ,Γ_K)-submodules of D, for $0\le j\le n$ we consider the extension

$$0\to\mathrm{Fil}^j(D)\to D\to D/\mathrm{Fil}^j(D)\to 0.$$

Let Σ_K be the set of embeddings from K to E. We always assume that $|\Sigma_K|=[K:\mathbb{Q}_p]$. For any $\mathbf{k}=(k_\sigma)_\sigma\in\mathbb{N}^{\Sigma_K}$, let $t^{\mathbf{k}}:=\prod_{\sigma:K\to E}t_\sigma^{k_\sigma}\in\mathcal{R}_{K,E}$, cf. [KPX14, Notation 6.2.7]. Then,[1] we can pullback D along $t^{\mathbf{k}}(D/\mathrm{Fil}^j(D))\subset D/\mathrm{Fil}^j(D)$ to get a subobject $p_{\mathbf{k}}(D,\mathrm{Fil}^j(D))$ of D:

$$0\to\mathrm{Fil}^j(D)\to p_{\mathbf{k}}(D,\mathrm{Fil}^j(D))\to t^{\mathbf{k}}(D/\mathrm{Fil}^j(D))\to 0.$$

Note that if $(h_{i,\sigma})_{1\le i\le n,\sigma}\in(A^n)^{\Sigma_K}$ are the Sen weights of D, with $(h_{i,\sigma})_\sigma\in A^{\Sigma_K}$ being the Sen weights of the (φ,Γ_K)-module $\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)$ of rank 1 for $1\le i\le n$, then

$$h'_{i,\sigma}=\begin{cases}h_{i,\sigma} & \text{if } i\le j\\ h_{i,\sigma}+k_\sigma & \text{if } i>j\end{cases}$$

are the Sen weights of $p_{\mathbf{k}}(D,\mathrm{Fil}^j(D))$ of D.

One can then ask whether these weight-shifting operations descend to the category of trianguline (φ,Γ_K)-modules (without fixing a filtration). But it is easy to see that in general they depend on the choice of triangulation, as the following example shows.

When $K=\mathbb{Q}_p$, for an integer i, we write $x^i:\mathbb{Q}_p^\times\to E^\times, a\mapsto a^i$, and denote by $R_{\mathbb{Q}_p,E}(x^i)$ the $(\varphi,\Gamma_{\mathbb{Q}_p})$-module of rank 1 associated to this character. Explicitly, it is the free $R_{\mathbb{Q}_p,E}$-module $R_{\mathbb{Q}_p,E}e_i$ of rank 1 with $\varphi(e_i)=p^ie_i$ and $\gamma(e_i)=\varepsilon(\gamma)^ie_i$ for $\gamma\in\Gamma_{\mathbb{Q}_p}$, where $\varepsilon:\Gamma_{\mathbb{Q}_p}\xrightarrow{\sim}\mathbb{Z}_p^\times$ is the

[1] As for the pushforward operations, starting from the extension $0\to\mathrm{Fil}^j(D)\to D\to D/\mathrm{Fil}^j(D)\to 0$, we may also consider its pushforward along the inclusion $\mathrm{Fil}^j(D)\subset t^{-\mathbf{k}}\mathrm{Fil}^j(D)$, resulting in a (φ,Γ_K)-submodule $\iota_{\mathbf{k}}(D)$ of $t^{-\mathbf{k}}D$ that is an extension of the form

$$0\to t^{-\mathbf{k}}\mathrm{Fil}^j(D)\to\iota_{\mathbf{k}}(D)\to D/\mathrm{Fil}^j(D)\to 0.$$

The pushforward $\iota_{\mathbf{k}}(D)$ becomes isomorphic to the pullback $p_{\mathbf{k}}(D)$ after a character twist (Lemma 2.2). Hence, we mainly focus on the pullback operations $p_{\mathbf{k}}$ in the article.

cyclotomic character. Then, $R_{\mathbb{Q}_p,E}(x^i)$ has Sen weight i. Let $D = \mathcal{R}_{\mathbb{Q}_p,E}(x) \oplus \mathcal{R}_{\mathbb{Q}_p,E}(x^2)$ be split[2] of rank 2 and Hodge-Tate-Sen weights $\{1,2\}$. Then we have two triangulations

$$0 \to \mathcal{R}_{\mathbb{Q}_p,E}(x) \to D \to \mathcal{R}_{\mathbb{Q}_p,E}(x^2) \to 0,$$
$$0 \to \mathcal{R}_{\mathbb{Q}_p,E}(x^2) \to D \to \mathcal{R}_{\mathbb{Q}_p,E}(x) \to 0.$$

Applying pullbacks with $k > 0$, we get $\mathcal{R}_{\mathbb{Q}_p,E}(x) \oplus \mathcal{R}_{\mathbb{Q}_p,E}(x^{k+2})$ and $\mathcal{R}_{\mathbb{Q}_p,E}(x^2) \oplus \mathcal{R}_{\mathbb{Q}_p,E}(x^{k+1})$ respectively, which are of different Hodge-Tate-Sen weights $\{k+2, 1\}$ and $\{k+1, 2\}$.

1.1.2. *Using Sen polynomials.* However, [Wu, Proposition 3.16] implies that this weight issue is the only obstruction for the pullback to be independent of the triangulation. Let $K = \mathbb{Q}_p$ here for simplicity, and let A be any affinoid E-algebra. Wu showed that if the Sen polynomial $P_{\mathrm{Sen},D}(T) \in A[T]$ of D admits a factorization

$$P_{\mathrm{Sen},D}(T) = Q(T)S(T)$$

with monic $Q(T), S(T) \in A[T]$, with $(Q, S) = 1$, then there exists a unique (φ, Γ)-submodule D' of D satisfying

$$tD \subset D' \subset D$$

and having Sen polynomial $Q(T-1)S(T)$. The proof used Beauville-Laszlo gluing [Wu, Proposition A.3] to show that, under the equivalence of (φ, Γ_K)-modules and Γ_K-equivariant vector bundles on the Fargues-Fontaine curve $X_{K_\infty,A}$ [EGH, Theorem 5.1.5], with respect to the canonical decomposition of the Sen module

$$D_{\mathrm{Sen}}(D) = \ker(Q(\Theta_{\mathrm{Sen}})) \oplus \ker(S(\Theta_{\mathrm{Sen}})),$$

there is a unique modification of the (φ, Γ)-module D at ∞ on the curve $X_{K_\infty,A}$ such that the resulting equivariant subbundle $D' \subset D$ has the Sen polynomial $P_{\mathrm{Sen},D'}(T) = Q(T-1)S(T)$.

If D is a trianguline (φ, Γ_K)-module over $\mathcal{R}_{K,A}$ and $\mathrm{Fil}^\bullet(D)$ is a triangulation on D, then for any $1 \le j \le n$, the extension

$$0 \to \mathrm{Fil}^j(D) \to D \to D/\mathrm{Fil}^j(D) \to 0$$

induces a splitting of the Sen polynomial

$$P_{\mathrm{Sen},D}(T) = P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T) \cdot P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T), \tag{1.1.1}$$

and $p_1(D, \mathrm{Fil}^j(D))$ is a (φ, Γ)-submodule of D satisfying

$$tD \subset p_1(D, \mathrm{Fil}^j(D)) \subset D$$

and having Sen polynomial $P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T-1)P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T)$. If (1.1.1) is a comaximal factorization, then $p_1(D, \mathrm{Fil}^j(D))$ is the unique (φ, Γ)-submodule of D containing tD of the Sen polynomial $P_{\mathrm{Sen},D/\mathrm{Fil}^j(D)}(T-1)P_{\mathrm{Sen},\mathrm{Fil}^j(D)}(T)$. Moreover, in this case, for any triangulation $\mathrm{Fil}^\bullet(D)'$ on D inducing the same splitting as (1.1.1), by the uniqueness we have that

$$p_1(D, \mathrm{Fil}^j(D)') = p_1(D, \mathrm{Fil}^j(D)).$$

Consider those (φ, Γ_K)-modules D on $\mathrm{Sp}(A)$ that are Tate-fpqc locally trianguline. Assume moreover that at each point $z \in \mathrm{Sp}(A)$, the fiber $D_z := D \otimes_A k(z)$ is "weight-uniform trianguline" in the sense that all possible triangulations on $D_z \otimes_{k(z)} L$ induce the same ordering on the set of Sen weights of D_z, for all finite extensions $L/k(z)$. Any local triangulation on D locally

[2]We refer the reader to Remark 3.6 for a non-split example.

gives us a factorization of the Sen polynomial. After imposing necessary restrictions on its Sen weights at geometric fibers, we can perform such "modifications at ∞" iteratively on D in an invertible way that is independent of the choice of covers and local triangulations, whenever the resulting movement of our ordered Sen weights in the weight space does not meet any of the relevant walls, cf. Theorem 4.10 for our exposition on Wu's result and Theorem 4.12 for discussion in the weight-uniform trianguline case.

1.1.3. *The point of view of the analytic Emerton-Gee stacks.* Recall from [EGH, §5.3] that over the category Rig_E of rigid analytic spaces over E equipped with the Tate-fpqc topology, we have the moduli stack $\mathfrak{X}_n$ of rank n G_K-equivariant vector bundles over the Fargues-Fontaine curve $X_{\overline{K}}$, and the stack $\mathfrak{X}_B$ of G_K-equivariant B-bundles on $X_{\overline{K}}$, where B denotes the Borel subgroup of $G = \mathrm{GL}_n$ consisting of upper triangular invertible matrices. Then, by the equivalence [EGH, Theorem 5.1.5], a triangulated (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ of rank n over $\mathcal{R}_{K,A}$ defines a point in $\mathfrak{X}_B(A)$, while the (φ, Γ_K)-module D defines a point in $\mathfrak{X}_n(A)$.

We introduce in Definition 4.4 "weight-uniform trianguline substack" $\mathfrak{X}_n^{\mathrm{wu}}$ of $\mathfrak{X}_n$, and some additional substacks $\mathfrak{X}_n^{\sigma\text{-wu},i} \subset \mathfrak{X}_n^{\sigma\text{-wu}}$ characterized by the property that, for any triangulation on D, the first $n-i$ σ-Sen weights are distinct from the last i σ-Sen weights at any $z \in \mathrm{Sp}(A)$. The pullback operator $p_{i,\sigma}$ (§4) that increases the last i σ-Sen weights by 1 and leaves all other Sen weights invariant descends to a map from $\mathfrak{X}_n^{\sigma\text{-wu},i}$ to $\mathfrak{X}_n$. Then, we deduce the following theorem, cf. Definition 4.6 (and Remark 4.7) for the precise meaning of our notation.

Theorem A. *The pullback maps $p_{i,\sigma} : \mathfrak{X}_B \to \mathfrak{X}_B$ descend to canonical morphisms of stacks*

$$p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i} \longrightarrow \mathfrak{X}_n$$

such that for $S \subset \Sigma_K$, $I = \prod_{\sigma \in S} I_\sigma \subset \{1, \dots, n\}^S$ and $\mathbf{k} = (k_{i,\sigma})_{\sigma \in S, i \in I_\sigma} \in \mathbb{N}^I$,

$$p_{\mathbf{k}} := \prod_{i,\sigma} (p_{i,\sigma})^{k_{i,\sigma}} : \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}} \xrightarrow{\sim} \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}$$

are isomorphisms between these weight-uniform trianguline substacks, where the change of Sen weights does not change the regularity of the weights.

1.1.4. We also mention that, for such directions of changing the weights, Wu obtained in [Wu, §3] general geometric results: the stack $\mathfrak{X}_n$ of rank n (φ, Γ_K)-modules at integral Hodge-Tate weights $\mathbf{h} = (h_{i,\sigma}) \in (\mathbb{Z}^n)^{\Sigma_K}$ are described using a "product formula" of the form (if $K = \mathbb{Q}_p$)

$$(\mathfrak{X}_n)^\wedge_{\mathbf{h}} \cong (\mathfrak{X})^\wedge_0 \times_{\mathfrak{g}/\mathrm{GL}_n} \widetilde{\mathfrak{g}}_{P_{\mathbf{h}}} / \mathrm{GL}_n,$$

which then induces change of weights maps

$$f_{\mathbf{h},\mathbf{h}'} : (\mathfrak{X}_n)^\wedge_{\mathbf{h}} \to (\mathfrak{X}_n)^\wedge_{\mathbf{h}'}$$

and this can be formulated for non-integral weights using local models developed in [Wu22, Ch. 5]. Moreover, change of weights maps $f_{\mathbf{h},\mathbf{h}'}$ exist at arbitrary weights $\mathbf{h}$ when changing from $\mathbf{h}$ to $\mathbf{h}'$ does not increase the regularity, cf. [Wu, §1.3] for a discussion and [Wu, §3.3] for details.

Question. What can be said about the geometry of $\mathfrak{X}_n^{\mathrm{wu}}$?

1.1.5. The change-of-weight operations are supposed to the kind of operations on the side of p-adic Galois representations (or rather (φ, Γ_K)-modules D) that correspond to the effect of translation functor (as initially introduced in [JLS24]) on the side of locally analytic $\mathrm{GL}_n(K)$-representations under a (hypothetical) p-adic locally analytic Langlands correspondence $D \mapsto \Pi^{\mathrm{an}}(D)$. Roughly speaking, the change-of-weight operations and the translation functors are meant to be related by a formula of the type

$$\Pi^{\mathrm{an}}((\text{change-of-weight operation})(D)) = (\text{translation functor})(\Pi^{\mathrm{an}}(D)), \tag{1.1.2}$$

which has been shown in certain cases by [JLS24] and [Dina] for $\mathrm{GL}_2(\mathbb{Q}_p)$. We also remark that there are analogous statements in the local Langlands correspondence for real reductive groups which relate operations on "L-parameters" (on the Galois side) with translation functors on the corresponding representations of the reductive group, cf. [ABV92, 16.6].[3]

1.1.6. *Relation with translation functor under Ding's crystabelline correspondence.* A class of points of $\mathfrak{X}_n^{\mathrm{wu}}(E)$ are those non-critical crystabelline (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$. Recently, for non-critical crystabelline (φ, Γ_K)-modules D over $\mathcal{R}_{K,E}$ of regular Hodge-Tate weights, Ding constructed in [Din25] locally $\mathbb{Q}_p$-analytic representations $\pi_{\min}(D) \subset \pi_{\mathrm{fs}}(D)$ of $\mathrm{GL}_n(K)$, which are expected to occur as closed subrepresentations of the hypothetical $\Pi^{\mathrm{an}}(D)$. They are constructed as extensions of locally algebraic representations $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by locally analytic representations $\pi(\phi, \mathbf{h})$ that only depend on $D[1/t]$ and the Hodge-Tate weights $\mathbf{h}$. If $K = \mathbb{Q}_p$, these extension classes can "recover the Hodge filtration" and hence determine D, cf. [Din25, Theorem 3.34]. The pullback D' is a non-critical crystabelline (φ, Γ_K)-submodule of D whenever this change of weights preserves the regularity. It is then natural to expect that under $\pi := \pi_{\min}$ or π_{fs}, pulling back from D to D' corresponds to translating from $\pi(D)$ to $\pi(D')$. We prove the expected intertwining (1.1.2) of the two kinds of weight-shifting operations:

Theorem B. *Let D be a non-critical crystabelline (φ, Γ_K)-module with regular Sen weights $\mathbf{h}$. Let $p_{\mathbf{k}}(D) = f_{\mathbf{h},\mathbf{h}'}(D)$ be the (φ, Γ_K)-submodule of D obtained by a sequence of pullback operators $p_{\mathbf{k}} = \prod_{i,\sigma}(p_{i,\sigma})^{k_{i,\sigma}}$ such that its weights $\mathbf{h}'$ are still regular. Let $\lambda' := \mathbf{h}' - \theta$ and $\lambda := \mathbf{h} - \theta$ be the "automorphic weights" with $\theta := (0, -1, \ldots, -(n-1))_{\sigma \in \Sigma_K} \in (\mathbb{Z}^n)^{\Sigma_K}$. Then,*

$$T_{\lambda}^{\lambda'}(\pi_{\bullet}(D)) = \pi_{\bullet}(p_{\mathbf{k}}(D)) = \pi_{\bullet}(f_{\mathbf{h},\mathbf{h}'}(D))$$

as locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_n(K)$, for $\bullet \in \{\min, \mathrm{fs}\}$.

Since translation functors between regular weights are equivalences of categories, Theorem B follows from various results in [Din25], [JLS24] and [Wu], with some additional work.

1.2. **Structure of the paper.** In §2, we recall basics of (φ, Γ_K)-modules over affinoid algebras and their cohomology, generalizing a pointwise result [Col08, Theorem 2.22(i)] to the affinoid coefficients in Lemma 2.9(i), leading to Theorem 3.1 for trianguline families. In §3, we review triangulations on generic crystabelline (φ, Γ_K)-modules over fields, and observe that "generically, a trianguline (φ, Γ_K)-module has a unique non-split triangulation." In §4, we discuss Wu's change of weights maps in general and in the weight-uniform trianguline families (cf. Theorem 4.10 = [Wu, Proposition 3.16], and Theorem 4.12 = Theorem A), which is followed by a discussion on when étaleness can be preserved up to twist for local Artinian E-algebras $A \in \mathcal{C}_E$ of residue field E. Finally, §5 is devoted to Theorem B.

[3]Note that in [ABV92, 16.6] the objects on the Galois side are rather complete geometric parameters and on the automorphic side these are called "final limit characters".

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2. Preliminaries

2.0.1. *Notations and conventions.* Let K be a finite extension of $\mathbb{Q}_p$ of ramification index e and inertial degree f, with maximal unramified subfield K_0 and a chosen uniformizer π_K. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K, W_K be the Weil group of K, $\mathrm{rec}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ be the local reciprocity map normalized by sending π_K to a geometric Frobenius element, and let Σ_K be the set of all $\mathbb{Q}_p$-algebra embeddings of K into $\overline{\mathbb{Q}_p}$. Let E be a finite extension of $\mathbb{Q}_p$ such that all the $\mathbb{Q}_p$-algebra embeddings of K into $\overline{\mathbb{Q}_p}$ factor through E, which will be our coefficient field. We allow E to be enlarged at will.

Let Rig_E be the category of rigid E-analytic spaces, and Aff_E the category of affinoid E-algebras. For $X \in \mathrm{Rig}_E$, let $\mathcal{R}_{K,X}$ be the relative Robba ring of K over X, and write $\mathcal{R}_{K,A} := \mathcal{R}_{K,\mathrm{Sp}(A)}$ for affinoid $A \in \mathrm{Aff}_E$. Let $\Gamma_K := \mathrm{Gal}(K(\mu_{p^\infty})|K)$. For a review of (generalized) (φ, Γ_K)-modules D over the Robba ring $\mathcal{R}_{K,X}$, we refer the reader to [Ber17, §2.1]. For a (φ, Γ_K)-module D on X, and $z \in X$ a rigid point, we write $D_z := D \otimes_{\mathcal{O}_X} k(z)$ for the fiber of D at z. Similarly, if $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$ is a locally $\mathbb{Q}_p$-analytic character, we write $\delta_z : K^\times \to \Gamma(X, \mathcal{O}_X)^\times \to k(z)^\times$ for the evaluation of δ at $z \in X$. The notation $D = (Q - P)$ is used to indicate that D is an extension of P by Q, and the notation $D = (Q_1 - \cdots - Q_r)$ indicates that there exists an increasing filtration $(\mathrm{Fil}^i(D))_{0\le i\le r}$ on D by subobjects such that $\mathrm{Fil}^0(D) = 0$, $\mathrm{Fil}^r(D) = D$, and $\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D) \cong Q_i$ for $1 \le i \le r$.

We set $\mathbb{N} := \mathbb{Z}_{\ge 0}$. By local class field theory, the p-adic cyclotomic character $\varepsilon : G_K \to \mathbb{Z}_p^\times$ is equivalently given by

$$\varepsilon : K^\times \longrightarrow E^\times, \quad x \mapsto N_{K|\mathbb{Q}_p}(x)|N_{K|\mathbb{Q}_p}(x)|_p\,,$$

which has σ-Sen weight $+1$ for all $\sigma \in \Sigma_K$. For a $[K : \mathbb{Q}_p]$-tuple of integers $\mathbf{k} = (k_\sigma)_{\sigma\in\Sigma_K} \in \mathbb{Z}^{\Sigma_K}$, set $t^{\mathbf{k}} := \prod_{\sigma\in\Sigma_K} t_\sigma^{k_\sigma} \in \mathcal{R}_{K,E}$, where $t_\sigma \in \mathcal{R}_{K,E}$ are the Lubin-Tate elements defined up to units in [KPX14, Notation 6.2.7]. Let x_σ be the embedding $\sigma : K^\times \to E^\times$ viewed as a character of $K^\times$, and set $x^{\mathbf{k}} := \prod_{\sigma\in\Sigma_K} x_\sigma^{k_\sigma}$. Then, the (φ, Γ_K)-module $t^{\mathbf{k}}\mathcal{R}_{K,A} \cong \mathcal{R}_{K,A}(x^{\mathbf{k}})$ is free of rank 1 with σ-Sen weight k_σ for each $\sigma \in \Sigma_K$.

2.0.2. *Extensions of (φ, Γ_K)-modules.* We recall certain operations on extensions of (φ, Γ_K)-modules and the induced linear maps on (φ, Γ_K)-cohomology.

Definition 2.1. For $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$ and (φ, Γ_K)-modules D_1, D_2 over $\mathcal{R}_{K,A}$, there are two natural A-linear maps between Yoneda Ext groups:

- pushing out along $D_1 \hookrightarrow t^{-\mathbf{k}}D_1$ defines a map
$$\iota_{\mathbf{k}} : \mathrm{Ext}^1(D_2, D_1) \longrightarrow \mathrm{Ext}^1(D_2, t^{-\mathbf{k}}D_1),$$
- pulling back along $t^{\mathbf{k}}D_2 \hookrightarrow D_2$ defines a map
$$p_{\mathbf{k}} : \mathrm{Ext}^1(D_2, D_1) \longrightarrow \mathrm{Ext}^1(t^{\mathbf{k}}D_2, D_1).$$

The following lemma shows that these two maps are related by the "twisting" isomorphism

$$x^{\mathbf{k}} : \mathrm{Ext}^1(D_2, t^{-\mathbf{k}}D_1) \xrightarrow{\simeq} \mathrm{Ext}^1(t^{\mathbf{k}}D_2, D_1)$$

induced by twisting by the character $x^{\mathbf{k}} : K^\times \to A^\times$, so that $x^{\mathbf{k}} \circ \iota_{\mathbf{k}} = p_{\mathbf{k}}$.

Lemma 2.2. *Let $\mathbf{k} = (k_\sigma)_\sigma \in \mathbb{N}^{\Sigma_K}$ and let*

$$0 \to D_1 \xrightarrow{i} D \xrightarrow{\pi} D_2 \to 0$$

be an exact sequence of (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$. Then, the pushout $\iota_{\mathbf{k}}(D)$ of D along $D_1 \hookrightarrow t^{-\mathbf{k}}D_1$ and the pullback $p_{\mathbf{k}}(D)$ of D along $t^{\mathbf{k}}D_2 \hookrightarrow D_2$ are related by a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t^{-\mathbf{k}}D_1 & \longrightarrow & \iota_{\mathbf{k}}(D) & \longrightarrow & D_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & D_1 & \xrightarrow{i} & D & \xrightarrow{\pi} & D_2 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D_1 & \longrightarrow & p_{\mathbf{k}}(D) & \longrightarrow & t^{\mathbf{k}}D_2 & \longrightarrow & 0
\end{array}$$

with exact rows and injective columns, from which we deduce $p_{\mathbf{k}}(D) = t^{\mathbf{k}}\iota_{\mathbf{k}}(D) \cong \iota_{\mathbf{k}}(D)(x^{\mathbf{k}})$.

Proof. Explicitly, $p_{\mathbf{k}}(D) = \pi^{-1}(t^{\mathbf{k}}D_2)$, and $\iota_{\mathbf{k}}(D) = (t^{-\mathbf{k}}D_1) \oplus_{D_1} D$ is an amalgamated sum over D_1. By the first row, we see that $t^{\mathbf{k}}\iota_{\mathbf{k}}(D)$ is a (φ, Γ_K)-submodule of $D[1/t]$ containing D_1 with the associated quotient being $t^{\mathbf{k}}D_2$, so it equals $p_{\mathbf{k}}(D)$ by the third row. □

By Lemma 2.2, the pullback $p_{\mathbf{k}}$ is injective/surjective/zero if and only if the pushout $\iota_{\mathbf{k}}$ is. We recall the following cohomological interpretation of the extensions.

Lemma 2.3. *Let D_1, D_2 be (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$. Then,*

$$H^1(D_2^\vee \otimes D_1) \cong \mathrm{Ext}^1(D_2, D_1)$$

where H^1 is computed using the Herr complex

$$\mathcal{C}^\bullet(D) : \quad D \xrightarrow{(\varphi-1,\gamma-1)} D \oplus D \xrightarrow{(\gamma-1)\oplus(1-\varphi)} D$$

for $D := D_2^\vee \otimes D_1 \cong \mathrm{Hom}_{\mathcal{R}_{K,A}}(D_2, D_1)$, where $\gamma \in \Gamma_K$ is a topological generator.

Proof. This is essentially [EG23, Lemma 5.1.2], but adapted to our convention and notation.

Let M be an extension of D_2 by D_1 as (φ, Γ_K)-modules over $\mathcal{R} := \mathcal{R}_{K,A}$ given by

$$0 \to D_1 \xrightarrow{i} M \xrightarrow{\pi} D_2 \to 0.$$

It splits on the level of $\mathcal{R}$-modules. We choose any $\mathcal{R}$-linear section $s : D_2 \to M$, which is unique up to an element $h \in D = \mathrm{Hom}_{\mathcal{R}}(D_2, D_1)$. Using the section s, we can write

$$\varphi_M = \begin{pmatrix} \varphi_{D_1} & f \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} \quad \text{and} \quad \gamma_M = \begin{pmatrix} \gamma_{D_1} & g \circ \gamma_{D_2} \\ & \gamma_{D_2} \end{pmatrix}$$

for uniquely determined $f, g \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1) = D$ since D_2 has an $\mathcal{R}$-basis in $\varphi(D_2)$.

That is, for any $x \in D_2$, we have

$$f(\varphi_{D_2}(x)) = (\varphi_M - s \circ \varphi_{D_2} \circ \pi)(s(x)) = \varphi_M(s(x)) - s(\varphi_{D_2}(x)) \in \ker(\pi) = D_1.$$

Similarly, since γ acts invertibly on $\mathcal{R}$, we have

$$g = (\gamma_M - s \circ \gamma_{D_2} \circ \pi) \circ s \circ \gamma_{D_2}^{-1} = \gamma_M \circ s \circ \gamma_{D_2}^{-1} - s \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1).$$

The commutativity $\varphi_M \circ \gamma_M = \gamma_M \circ \varphi_M$ is then equivalent to the equality

$$\varphi_{D_1} g \gamma_{D_2} + f \varphi_{D_2} \gamma_{D_2} = \gamma_{D_1} f \varphi_{D_2} + g \gamma_{D_2} \varphi_{D_2} \in \mathrm{Hom}_{\mathcal{R}}(D_2, D_1)$$

which, by precomposing with $\gamma_{D_2}^{-1}$ on D_2, is equivalent to

$$\varphi_{D_1} g - g \varphi_{D_2} = \gamma_{D_1} f \gamma_{D_2}^{-1} \varphi_{D_2} - f \varphi_{D_2}.$$

By the definition of $\mathrm{Hom}_{\mathcal{R}}(D_1, D_2)$ as (φ, Γ_K)-module, this last displayed equation is the same as $(\varphi_D - 1).g = (\gamma_D - 1).f \in D$, hence $(f, g) \in \ker((\gamma - 1) \oplus (1 - \varphi))$ is a 1-cocycle.

Modifying the section s by $h \in D$ results in another 1-cocycle (f', g') such that

$$\begin{pmatrix} \varphi_{D_1} & f' \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} = \begin{pmatrix} 1 & -h \\ & 1 \end{pmatrix} \begin{pmatrix} \varphi_{D_1} & f \circ \varphi_{D_2} \\ & \varphi_{D_2} \end{pmatrix} \begin{pmatrix} 1 & h \\ & 1 \end{pmatrix}$$

(and a similar identity involving g, g', γ_{D_1} and γ_{D_2}) which is equivalent to

$$f' \varphi_{D_2} = f \varphi_{D_2} - h \varphi_{D_2} + \varphi_{D_1} h \quad (\text{resp. } g' = g - h + \gamma_{D_1} h \gamma_{D_2}^{-1}).$$

Thus, $f' = f + (\varphi_D - 1).h$ and $g' = g + (\gamma_D - 1).h$. So, $(f', g') - (f, g)$ is a 1-coboundary. □

Given Lemma 2.3, we see that $\iota_{\mathbf{k}}$ and $p_{\mathbf{k}}$ induce the same map

$$H^1(D) \longrightarrow H^1(t^{-\mathbf{k}} D)$$

for $D := D_2^\vee \otimes D_1$, if we identify the codomains $H^1(D_2^\vee \otimes t^{-\mathbf{k}} D_1) \simeq H^1((t^{\mathbf{k}} D_2)^\vee \otimes D_1)$ via $x^{\mathbf{k}}$.

Example 2.4. When $D_2 = \mathcal{R}_{\mathbb{Q}_p}$, Lemma 2.3 was proven by Colmez [Col08, §2.1]: take $D_2 = \mathcal{R}$ and $D_1 = \mathcal{R}(\delta)$; then the isomorphism

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}, \mathcal{R}(\delta)) \xrightarrow{\simeq} H^1(\delta)$$

given in [Col08, §2.1] is as follows: given an extension M, let $e \in M$ be a lift of the vector $1 \in \mathcal{R}$, and we associate to M (the class of) the 1-cocycle $[(\varphi_M - 1)e, (\gamma_M - 1)e] \in R(\delta) \oplus R(\delta)$. This is the map constructed in Lemma 2.3. Indeed, we have $s : 1 \mapsto e$, $\varphi_{D_2} = \gamma_{D_2} = \mathrm{id}$, so we deduce $f = \varphi_M(e) - e = (\varphi_M - 1)e$. Likewise, $g = (\gamma_M - 1)e$, as desired.

2.0.3. We review some results on (φ, Γ_K)-modules of rank 1 and their cohomology.

Theorem 2.5. *For any (φ, Γ_K)-module D of rank 1 over a rigid analytic space X, there exist a unique continuous character $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and a unique up to isomorphism line bundle $\mathcal{L}$ on X such that*

$$\mathcal{R}_{K,X}(\delta) \otimes_{\mathcal{O}_X} \mathcal{L} \cong D.$$

In fact, the line bundle $\mathcal{L}$ is given by $\mathcal{L} = H^0_{\varphi,\Gamma_K}(D(\delta^{-1})) \cong \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,X}(\delta), D)$, and the canonical map

$$\mathcal{R}_{K,X}(\delta) \otimes_{\mathcal{O}_X} H^0_{\varphi,\Gamma_K}(D(\delta^{-1})) \longrightarrow D$$

is an isomorphism.

Proof. This is [KPX14, Theorem 6.2.14]. For the definition of the (φ, Γ_K)-module $\mathcal{R}_{K,X}(\delta)$ of rank 1, see [KPX14, Construction 6.2.4]. □

Definition 2.6. Let A be an affinoid E-algebra. Let $\delta : K^\times \to A^\times$ be a continuous character, which is automatically locally $\mathbb{Q}_p$-analytic by [Buz07, Proposition 8.3]. Then, its derivative at 1 defines a $\mathbb{Q}_p$-linear map $d\delta : K \to A$, and hence an A-linear map $K \otimes_{\mathbb{Q}_p} A \to A$. Via the map

$$K \otimes_{\mathbb{Q}_p} A \xrightarrow{\sim} \prod_{\sigma\in\Sigma_K} A, \quad x \otimes y \mapsto (\sigma(x)y)_{\sigma\in\Sigma_K}, \tag{2.0.1}$$

we have an isomorphism

$$\mathrm{Hom}_{\mathbb{Q}_p}(K, A) \cong \mathrm{Hom}_A(K \otimes_{\mathbb{Q}_p} A, A) \cong \mathrm{Hom}_A(\prod_{\sigma\in\Sigma_K} A, A) = A^{\Sigma_K}.$$

The image of $d\delta$ is a $[K : \mathbb{Q}_p]$-tuple $\mathrm{wt}(\delta) := (\mathrm{wt}_\sigma(\delta))_{\sigma\in\Sigma_K} \in A^{\Sigma_K}$, which we call the **weight** of the character δ. We also call $\mathrm{wt}_\sigma(\delta)$ the **σ-weight** of δ. By [KPX14, Lemma 6.2.12], for any continuous character $\delta : K^\times \to A^\times$, the ($\sigma$-)Sen weight of $\mathcal{R}_{K,A}(\delta)$ is the (σ-)weight of δ.

Lemma 2.7. *Let $\delta : K^\times \to E^\times$ be a continuous character.*

(i) For $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$, if $\mathrm{wt}_\sigma(\delta) \notin \{1, \dots, k_\sigma\}$ for each $\sigma \in \Sigma_K$, then

$$\iota_{\mathbf{k}} : H^1(\delta) \to H^1(x^{-\mathbf{k}}\delta)$$

is an isomorphism.

(ii) We have

- $\dim_E H^0(\delta) = \begin{cases} 1 & \text{if } \delta = x^{-\mathbf{k}} \text{ for some } \mathbf{k} \in \mathbb{N}^{\Sigma_K}, \\ 0 & \text{otherwise.} \end{cases}$
- $\dim_E H^2(\delta) = \begin{cases} 1 & \text{if } \delta = \left(N_{K|\mathbb{Q}_p}|N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \text{ for some } \mathbf{k} \in \mathbb{N}^{\Sigma_K}, \\ 0 & \text{otherwise.} \end{cases}$
- $\dim_E H^1(\delta) = \begin{cases} [K : \mathbb{Q}_p] + 1 & \text{if either } H^0 \text{ or } H^2 \text{ does not vanish,} \\ [K : \mathbb{Q}_p] & \text{otherwise.} \end{cases}$

(iii) Any nonzero (φ, Γ_K)-submodule of $\mathcal{R}_{K,E}(\delta)$ must be of the form $t^{\mathbf{k}}\mathcal{R}_{K,E}(\delta)$ for some $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$.

Proof. The first statement is [BHS19, Lemma 3.3.3], the second statement is [KPX14, Proposition 6.2.8], and the third statement is [KPX14, Corollary 6.2.9]. □

Definition 2.8. Let $\mathcal{T}$ denote the rigid analytic space of continuous characters of $K^\times$. Let $\mathcal{T}_{\mathrm{wreg}}$ denote the open[4] complement in $\mathcal{T}$ to the points

$$\left\{ \left(N_{K|\mathbb{Q}_p}|N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{\Sigma_K} \right\},$$

and let $\mathcal{T}_{\mathrm{reg}}$ denote the open complement in $\mathcal{T}$ to the points

$$\left\{ x^{-\mathbf{k}}, \left(N_{K|\mathbb{Q}_p}|N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{\Sigma_K} \right\}.$$

[4]Observe that any subset $\mathcal{Z}$ of $\left\{ x^{-\mathbf{k}}, \left(N_{K|\mathbb{Q}_p}|N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{\Sigma_K} \right\}$ is closed in $\mathcal{T}$ since $\mathcal{Z}$ is *locally finite*, i.e., for any affinoid subdomain $U = \mathrm{Sp}(A)$ of $\mathcal{T}$, $U \cap \mathcal{Z}$ is a finite set of rigid points, and hence closed. If $\mathcal{Z}$ is finite, then there is nothing to show. Assume that $\mathcal{Z}$ is infinite. Evaluation at $\pi_K \in K^\times$ of the universal character $\delta_A^{\mathrm{univ}} : K^\times \to \Gamma(U, \mathcal{O}_U)^\times = A^\times$ gives rise to a unit $f := \delta_A^{\mathrm{univ}}(\pi_K) \in A^\times$. By the maximum principle [Bos14, Theorem 3.1/15] applied to both f^{-1} and f, there exists $C > 0$ such that $C^{-1} \le |f(z)| \le C$ for every rigid point $z \in \mathrm{Sp}(A)$. However, since $\mathcal{Z}$ is infinite, the p-adic valuations of its elements evaluated at π_K are clearly unbounded. So, $U \cap \mathcal{Z}$ is a finite subset of $\mathcal{Z}$, and $\mathcal{Z}$ is locally finite.

For $n \geq 1$, we denote by $\mathcal{T}^n_{\mathrm{wreg}}$ (resp. $\mathcal{T}^n_{\mathrm{reg}}$) the open subspace of $\mathcal{T}^n$ consisting of characters $\delta = (\delta_1, \ldots, \delta_n)$ such that $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{wreg}}$ (resp. $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}}$) for all $1 \leq i \neq j \leq n$.

Lemma 2.9. *Let $\delta : K^\times \to A^\times$ be a continuous character. For $z \in \mathrm{Sp}(A)$ corresponding to a maximal ideal $\mathfrak{m}_z \subset A$, let $\delta_z : K^\times \to A^\times \twoheadrightarrow k(z)^\times$ denote the mod-$\mathfrak{m}_z$ reduction of δ.*

(i) For $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$, if $\mathrm{wt}_\sigma(\delta_z) \notin \{1, \ldots, k_\sigma\}$ for each $\sigma \in \Sigma_K$ and for all $z \in \mathrm{Sp}(A)$, then

$$\iota_{\mathbf{k}} : H^1(\delta) \to H^1(x^{-\mathbf{k}}\delta)$$

is an isomorphism.

(ii) If $\delta \in \mathcal{T}_{\mathrm{wreg}}(A)$, then $H^2(\delta) = 0$, and $H^1(\delta) \otimes_A k(z) \cong H^1(\delta_z)$ for all $z \in \mathrm{Sp}(A)$.

(iii) If $\delta \in \mathcal{T}_{\mathrm{reg}}(A)$, then $H^1(\delta)$ is locally free of rank $[K : \mathbb{Q}_p]$ over A and $H^0(\delta) = H^2(\delta) = 0$. Moreover, $H^i(\delta) \otimes_A k(z) \cong H^i(\delta_z)$ for all $i \in \{0, 1, 2\}$ and $z \in \mathrm{Sp}(A)$.

Proof. By [KPX14, Corollary 6.3.3], for any (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ and any $\sigma \in \Sigma_K$, $H^i(D)$ and $H^i(D/t_\sigma)$ are finitely generated A-modules; moreover, they are the cohomology of complexes of finite projective A-modules concentrated in degrees $[0, 2]$. So, as in [Stacks, Tag 061Z], we have the base change spectral sequence

$$E_2^{j,-i} = \mathrm{Tor}_i^A \left(H^j_{\varphi,\Gamma}(\heartsuit), k(z) \right) \Rightarrow H^{j-i}_{\varphi,\Gamma}(\heartsuit \otimes_A k(z)) \tag{2.0.2}$$

for $\heartsuit \in \{D, D/t_\sigma\}$. As the (φ, Γ_K)-cohomology are concentrated in $[0, 2]$, from (2.0.2) we see

$$H^2(\heartsuit) \otimes_A k(z) \xrightarrow{\sim} H^2(\heartsuit_z) \tag{2.0.3}$$

is an isomorphism for every $z \in \mathrm{Sp}(A)$.

(i) Since $\iota_{\mathbf{k}}$ is induced by $\mathcal{R}_{K,A}(\delta) \hookrightarrow t^{-k}\mathcal{R}_{K,A}(\delta) = \mathcal{R}_{K,A}(x^{-\mathbf{k}}\delta)$, which factors as

$$\mathcal{R}_{K,A} \subset t_{\sigma_1}^{-1}\mathcal{R}_{K,A} \subset \cdots \subset t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset t_{\sigma_2}^{-1}t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset \cdots \subset t_{\sigma_2}^{-k_{\sigma_2}}t_{\sigma_1}^{-k_{\sigma_1}}\mathcal{R}_{K,A} \subset \cdots \subset t^{-\mathbf{k}}\mathcal{R}_{K,A}$$

where we enumerate $\Sigma_K = \{\sigma_1, \sigma_2, \ldots, \sigma_{[K:\mathbb{Q}_p]}\}$, it suffices to prove that if $\mathrm{wt}_\sigma(\delta_z) \neq 1$ for all $z \in \mathrm{Sp}(A)$, then $\iota_\sigma : H^1(\delta) \to H^1(x_\sigma^{-1}\delta)$ is an isomorphism.

By the long exact sequence in cohomology attached to the exact sequence

$$0 \to \mathcal{R}_{K,A}(\delta) \to t_\sigma^{-1}\mathcal{R}_{K,A}(\delta) \to t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta) \to 0,$$

it suffices to show the vanishing of H^0 and H^1 of $t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)$.

For each $z \in \mathrm{Sp}(A)$, the fiber $t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z)$ has vanishing H^2 by [Liu08, Theorem 3.7(ii)] since it is a torsion (φ, Γ_K)-module. By base change (2.0.3) and that

$$t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta) \cong \mathcal{R}_{K,A}(\delta)/t_\sigma\mathcal{R}_{K,A}(\delta)$$

has coherent cohomology, we have $H^2(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) = 0$ by Nakayama's lemma. From the spectral sequence (2.0.2), we see that for all $z \in \mathrm{Sp}(A)$

$$H^1(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) \otimes_A k(z) \cong H^1(t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z))$$

which is zero since $\dim_{k(z)} H^0(t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z)) = 0$ by [Nak09, Lemma 2.16] and $\dim_{k(z)} H^0(t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z)) = \dim_{k(z)} H^1(t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z))$ by Euler-Poincare formula [Liu08, Theorem 4.3] and vanishing of H^2. Again by Nakayama's lemma, we deduce the vanishing of H^1 and hence from the spectral sequence (2.0.2)

$$H^0(t_\sigma^{-1}\mathcal{R}_{K,A}(\delta)/\mathcal{R}_{K,A}(\delta)) \otimes_A k(z) \cong H^0(t_\sigma^{-1}\mathcal{R}_{K,k(z)}(\delta_z)/\mathcal{R}_{K,k(z)}(\delta_z))$$

from which we deduce the vanishing of H^0 by Nakayama's lemma.

(ii) For $\delta \in \mathcal{T}_{\mathrm{wreg}}(A)$, $\delta_z : K^\times \to k(z)^\times$ does not belong to $\left\{\left(N_{K|\mathbb{Q}_p}|N_{K|\mathbb{Q}_p}|_p\right) x^{\mathbf{k}} \middle| \mathbf{k} \in \mathbb{N}^{\Sigma_K}\right\}$ for every $z \in \mathrm{Sp}(A)$, which implies that $H^2(\delta_z) = H^2(\delta) \otimes_A k(z) = 0$ for each $z \in \mathrm{Sp}(A)$ by (2.0.3) and Lemma 2.7(ii). By Nakayama's lemma, $H^2(\delta) = 0$. Hence, (2.0.2) shows

$$H^1(\mathcal{R}_{K,A}(\delta)) \otimes_A k(z) \cong H^1(\mathcal{R}_{K,k(z)}(\delta_z))$$

for all $z \in \mathrm{Sp}(A)$.

(iii) This is [HS16, Proposition 2.3]. □

2.0.4. Let D be a (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,A}$ for affinoid A equipped with a filtration

$$\mathrm{Fil}^\bullet(D): \quad \mathrm{Fil}^0(D) = 0 \subsetneq \mathrm{Fil}^1(D) \subsetneq \cdots \subsetneq \mathrm{Fil}^n(D) = D$$

by saturated (φ, Γ_K)-submodules $\mathrm{Fil}^i(D)$ such that $\mathrm{gr}^i(\mathrm{Fil}^\bullet(D)) := \mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)$ is locally free of rank 1 over $\mathcal{R}_{K,A}$ for $1 \le i \le d$. By Theorem 2.5, there exists a unique continuous character $\delta_i : K^\times \to A^\times$ such that for the line bundle $\mathcal{L}_i := \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_i), \mathrm{gr}^i(\mathrm{Fil}^\bullet(D)))$ the canonical map

$$\mathcal{R}_{K,A}(\delta_i) \otimes_A \mathcal{L}_i \longrightarrow \mathrm{gr}^i(\mathrm{Fil}^\bullet(D))$$

is an isomorphism. In this case, we say D is **trianguline**, the filtration $\mathrm{Fil}^\bullet(D)$ is a **triangulation** of D, and $\delta = (\delta_i)_{1 \le i \le n}$ is the **parameter** of $(D, \mathrm{Fil}^\bullet(D))$.

If the triangulation on D is clear, we write $D_i := \mathrm{Fil}^i(D)$ and $D^i := D/\mathrm{Fil}^{n-i}(D)$ so that the ranks of the subobject D_i and the quotient D^i are always i, for $1 \le i \le n$.

2.0.5. We discuss various notions of "non-split" for (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$.

Definition 2.10. (i) Given a (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ with triangulation $\mathrm{Fil}^\bullet(D)$, if[5]

$$0 \to \mathrm{Fil}^{i-1}(D_z) \to \mathrm{Fil}^i(D_z) \to \mathcal{R}(\delta_{i,z}) \to 0 \tag{2.0.4}$$

are non-split as (φ, Γ_K)-modules for all $2 \le i \le n$ and all rigid points z of $\mathrm{Sp}(A)$, then $(D, \mathrm{Fil}^\bullet(D))$ is called **non-split**.

(ii) Given a (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$ with triangulation $\mathrm{Fil}^\bullet(D)$, if the exact sequences

$$0 \to \mathcal{R}(\delta_{i-1,z}) \cong \mathrm{Fil}^{i-1}(D_z)/\mathrm{Fil}^{i-2}(D_z) \to \mathrm{Fil}^i(D_z)/\mathrm{Fil}^{i-2}(D_z) \to \mathcal{R}(\delta_{i,z}) \to 0 \tag{2.0.5}$$

are non-split as (φ, Γ_K)-modules for all $2 \le i \le n$ and all rigid points z of $\mathrm{Sp}(A)$, then $(D, \mathrm{Fil}^\bullet(D))$ is called **strongly non-split**.

If $(D, \mathrm{Fil}^\bullet(D))$ is strongly non-split, then it is non-split, cf. Remark 2.11(i). But the converse is false when the rank of D is at least 3, and a rank-3 counterexample is given in Remark 3.6.

Remark 2.11. (i) If $(D, \mathrm{Fil}^\bullet(D))$ is non-split (2.0.4) or strongly non-split (2.0.5), then

$$0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$$

is a non-split extension of (φ, Γ_K)-modules, for all $1 \le i \le n-1$.

(ii) If $D \in \mathrm{Ext}^1(D_2, D_1)$ is a non-split extension of trianguline (φ, Γ_K)-modules $(D_i, \mathrm{Fil}^\bullet(D_i))$ of rank r_i for $i = 1, 2$, then $(D, \mathrm{Fil}^\bullet(D))$ need *not* be non-split in the sense of (2.0.4), where $\mathrm{Fil}^\bullet(D)$ is the triangulation on D induced by $\mathrm{Fil}^\bullet(D_1)$ and $\mathrm{Fil}^\bullet(D_2)$.

[5] In this definition, we do not need to write the line bundles $\mathcal{L}_i \otimes_A k(z)$ because there are no nontrivial line bundles over the point $\mathrm{Sp}(k(z))$.

Proof. (i) The image of

$$0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$$

under the linear map

$$\mathrm{Ext}^1(D/\mathrm{Fil}^i(D), \mathrm{Fil}^i(D)) \xrightarrow[\text{pullback}]{\mathrm{Fil}^{i+1}/\mathrm{Fil}^i(D) \subset D/\mathrm{Fil}^i(D)} \mathrm{Ext}^1(\delta_{i+1}, \mathrm{Fil}^i(D))$$

is the extension $0 \to \mathrm{Fil}^i(D) \to \mathrm{Fil}^{i+1}(D) \to \mathcal{R}(\delta_{i+1}) \to 0$ in (2.0.4). So,

$$0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$$

is non-split if $(D, \mathrm{Fil}^\bullet(D))$ is non-split. Moreover, the image of this pullback extension

$$0 \to \mathrm{Fil}^i(D) \to \mathrm{Fil}^{i+1}(D) \to \mathcal{R}(\delta_{i+1}) \to 0$$

under the linear map

$$\mathrm{Ext}^1(\delta_{i+1}, \mathrm{Fil}^i(D)) \xrightarrow[\text{pushforward}]{\mathrm{Fil}^i(D) \twoheadrightarrow \mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)} \mathrm{Ext}^1(\delta_{i+1}, \delta_i)$$

is the extension $0 \to \mathcal{R}(\delta_i) \to \mathrm{Fil}^{i+1}(D)/\mathrm{Fil}^{i-1}(D) \to \mathcal{R}(\delta_{i+1}) \to 0$ in (2.0.5). So, we see that "strongly non-split" implies "non-split", and that

$$0 \to \mathrm{Fil}^i(D) \to D \to D/\mathrm{Fil}^i(D) \to 0$$

is non-split if $(D, \mathrm{Fil}^\bullet(D))$ is strongly non-split.

(ii) We give a counterexample for $n = 3$ and $\mathcal{R} = \mathcal{R}_{\mathbb{Q}_p,E}$. Consider $D_1 = \mathcal{R}(\delta_1)$ and $D_2 = (\mathcal{R}(\delta_2) - \mathcal{R}(\delta_3))$ the unique non-split extension with $\delta_1 = x$, $\delta_2 = |x|x$ and $\delta_3 = 1$. From the short exact sequence

$$0 \to \mathcal{R}(\delta_2) \to D_2 \to \mathcal{R}(\delta_3) \to 0,$$

we consider a segment of the associated long exact sequence

$$H^0_{\varphi,\Gamma}(\delta_1\delta_2^{-1}) \to H^1_{\varphi,\Gamma}(\delta_1\delta_3^{-1}) \to H^1_{\varphi,\Gamma}(D_2^\vee(\delta_1)) \to H^1_{\varphi,\Gamma}(\delta_1\delta_2^{-1}) \to H^2_{\varphi,\Gamma}(\delta_1\delta_3^{-1}),$$

and the maps between the H^1's are pullbacks. By Lemma 2.7(ii), this sequence becomes $0 \to E \to E^2 \to E \to 0$, which in concrete terms means that if we take the non-split extension of $\mathcal{R}(\delta_3)$ by $\mathcal{R}(\delta_1)$, and pullback along $D_2 \twoheadrightarrow \mathcal{R}(\delta_3)$, then we get a non-split extension D of $\mathcal{R}(\delta_3)$ by D_1 whose pullback along $\mathcal{R}(\delta_2) \hookrightarrow D_2$ is a split extension of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$, which is $\mathrm{Fil}^2(D)$. Hence, although D is non-split as an extension of D_2 by D_1, it is not non-split in the sense of (2.0.4). ∎

3. Triangulations on (φ, Γ_K)-modules

Theorem 3.1. *Let $(D, \mathrm{Fil}^\bullet(D))$ be trianguline with parameters $(\delta_1, \ldots, \delta_n) : (K^\times)^n \to A^\times$ over $\mathcal{R}_{K,A}$. For any fixed $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$ and $i \in \{1, \ldots, n\}$, suppose that*

$$\mathrm{wt}_\sigma(\delta_{j,z}/\delta_{\ell,z}) \notin \{1, \ldots, k_\sigma\}$$

for all $1 \le j \le n-i$, $n-i+1 \le \ell \le n$, for all $\sigma \in \Sigma_K$ and for all $z \in \mathrm{Sp}(A)$. Then

$$p_{\mathbf{k}} : \mathrm{Ext}^1(D/\mathrm{Fil}^{n-i}(D), \mathrm{Fil}^{n-i}(D)) \longrightarrow \mathrm{Ext}^1(t^{\mathbf{k}}(D/\mathrm{Fil}^{n-i}(D)), \mathrm{Fil}^{n-i}(D))$$

is an isomorphism.

Proof. By induction on $\mathbf{k}$ it suffices to prove the statement for a fixed $\sigma \in \Sigma_K$ with $k_\sigma = 1$ and $k_\tau = 0$ for all $\tau \in \Sigma_K \setminus \{\sigma\}$, under the assumption that $\mathrm{wt}_\sigma(\delta_{j,z}/\delta_{\ell,z}) \neq 1$ for $j \leq n-i < \ell$.

Recall our notation that $D_{n-i} := \mathrm{Fil}^{n-i}(D)$ and $D^i := D/\mathrm{Fil}^{n-i}(D)$. By Theorem 2.5, D is a successive extension of rank 1 (φ, Γ_K)-modules $\mathrm{gr}_m(\mathrm{Fil}^\bullet(D)) \cong \mathcal{R}_{K,A}(\delta_m)\hat{\otimes}_A \mathcal{L}_m$ for some line bundles $\mathcal{L}_m$ over A. There is a finite admissible cover $\{\mathrm{Sp}(A_\nu)\}_{\nu=1}^r$ of $\mathrm{Sp}(A)$ trivializing them. Since H^*_{φ,Γ_K} commutes with flat base change by [KPX14, Theorem 4.4.3(2)], passing to $\mathrm{Sp}(A_\nu)$ we may assume that the line bundles $\mathcal{L}_m$ are all trivial.

After the reductions above, we show that

$$p_{i,\sigma} : \mathrm{Ext}^1(D^i, D_{n-i}) \cong H^1(D^{i\vee} \otimes D_{n-i}) \to \mathrm{Ext}^1(t_\sigma D^i, D_{n-i}) \cong H^1(t_\sigma^{-1}(D^{i\vee} \otimes D_{n-i}))$$

is bijective, where we have by assumption that

$$D_{n-i} = (\mathcal{R}_{K,A}(\delta_1) - \cdots - \mathcal{R}_{K,A}(\delta_{n-i})) \quad \text{and} \quad D^i = (\mathcal{R}_{K,A}(\delta_{n-i+1}) - \cdots - \mathcal{R}_{K,A}(\delta_n)).$$

Thus, putting $M := D^{i\vee} \otimes D_{n-i}$, we see that M is trianguline over $\mathcal{R}_{K,A}$ with parameters

$$\left\{\delta_j\delta_\ell^{-1} \middle| 1 \leq j \leq n-i \text{ and } n-i+1 \leq \ell \leq n\right\}.$$

Note that $\mathrm{wt}_\sigma(\delta_j\delta_\ell^{-1}) \neq 1$ for these parameters by the assumption. By the short exact sequence

$$0 \to M \to t_\sigma^{-1}M \to t_\sigma^{-1}M/M \to 0,$$

it is enough to establish the vanishing of $H^i(t_\sigma^{-1}M/M)$ as coherent sheaves for $i = 0, 1, 2$.

This follows from a dévissage argument on the above triangulation on M, using the proof of Lemma 2.9(i). Indeed, we know that $M = (\mathcal{R}_{K,A}(\eta_1) - \cdots - \mathcal{R}_{K,A}(\eta_m))$ of parameters η_s whose weights are not 1 at all $z \in \mathrm{Sp}(A)$, so that

$$t_\sigma^{-1}M/M \cong M/t_\sigma = (\mathcal{R}_{K,A}(\eta_1)/t_\sigma - \cdots - \mathcal{R}_{K,A}(\eta_m)/t_\sigma).$$

We induct on the rank of M to show the vanishing of cohomologies of M/t_σ. The base case where $m = 1$ was shown in the proof of Lemma 2.9(i). For $m \geq 2$, we have

$$\cdots \to H^i(\mathcal{R}_{K,A}(\eta_1)/t_\sigma) \to H^i(M/t_\sigma) \to H^i((M/\mathcal{R}_{K,A}(\eta_1))/t_\sigma) \to \ldots$$

from which we know $H^i(M/t_\sigma) = 0$ by the induction hypothesis. □

3.1. **Very generic case.**

Definition 3.2. Let $(D, \mathrm{Fil}^\bullet(D))$ be a trianguline (φ, Γ)-module over $\mathcal{R}_{K,E}$ with parameter $\delta = (\delta_1, \ldots, \delta_n)$ and Sen weights $(h_{i,\sigma})_{i,\sigma} \in (E^n)^{\Sigma_K}$, where $h_{i,\sigma} := \mathrm{wt}_\sigma(\delta_i)$. We say that D is **very generic** if for each $\sigma \in \Sigma_K$, $h_{i,\sigma} - h_{j,\sigma} \notin \mathbb{Z}$ for all $i \neq j$.

Definition 3.3. Let $\mathcal{T}_\circ^n$ be the open subspace of $\mathcal{T}^n$ such that for $A \in \mathrm{Aff}_E$, $\mathcal{T}_\circ^n(A)$ is the set of all the continuous $A^\times$-valued characters of $(K^\times)^n$

$$\delta = (\delta_1, \ldots, \delta_n) : (K^\times)^n \to A^\times$$

satisfying $\mathrm{wt}_\sigma(\delta_{i,z}/\delta_{j,z}) \notin \mathbb{Z}_{\geq 1}$ for all $1 \leq i < j \leq n$, all $\sigma \in \Sigma_K$, and all $z \in \mathrm{Sp}(A)$.

Proposition 3.4. *For trianguline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ of parameters $\delta = (\delta_i)_{1\leq i\leq n} \in \mathcal{T}_\circ^n(E)$ over $\mathcal{R}_{K,E}$ and a fixed index $i \in \{1, \ldots, n\}$, suppose*

(a). $\mathrm{Fil}^{n-i}(D)$ with the induced triangulation is strongly non-split, and

(b). $D/\mathrm{Fil}^{n-i-1}(D)$ with the induced triangulation is non-split.

Then, for each $1 \leq j \leq n-i$, $\mathrm{Fil}^j(D)$ is the unique saturated (φ, Γ_K)-submodule of D of rank j.

Proof. Note that similar to Remark 2.11, (a) and (b) imply that $(D, \mathrm{Fil}^\bullet(D))$ is non-split.

We proceed by induction. Let D' be a saturated (φ, Γ_K)-submodule of D. We claim that D' contains $\mathrm{Fil}^1(D)$. Indeed, let m be the largest integer such that $\mathrm{Fil}^{m-1}(D) \cap D' = 0$. Then $D' \cap \mathrm{Fil}^m(D)$ is an $\mathcal{R}_{K,E}$-submodule of D' stable under the (φ, Γ_K)-action such that the quotient $D'/(D' \cap \mathrm{Fil}^m(D))$ is torsion-free as it injects into $D/\mathrm{Fil}^m(D)$. As

$$\mathcal{R}_{K,E} \simeq \prod_{\tau: K_0 \hookrightarrow E} \mathcal{R}_{K,\mathbb{Q}_p} \otimes_{K_0,\tau} E$$

is a product of Bézout domains on which φ acts transitively, it follows from the torsion-freeness that $D' \cap \mathrm{Fil}^m(D)$ is free over $\mathcal{R}_{K,E}$ (cf. [Ber17, §2.6]) and hence is a (φ, Γ_K)-module.

Now if $m > 1$, then $D' \cap \mathrm{Fil}^m(D)$ is a nonzero (φ, Γ_K)-submodule of $\mathrm{Fil}^m(D)$ injecting to the quotient $\mathrm{Fil}^m(D)/\mathrm{Fil}^{m-1}(D) \cong \mathcal{R}_{K,E}(\delta_m)$, which is of the form $t^{\mathbf{k}}\mathcal{R}(\delta_m) = \mathcal{R}_{K,E}(x^{\mathbf{k}}\delta_m)$ for some $\mathbf{k} \in \mathbb{N}^{\Sigma_K}$ by Lemma 2.7(iii). Then, $\mathrm{Fil}^m(D)$ contains the split (φ, Γ_K)-submodule

$$\mathrm{Fil}^{m-1}(D) \oplus (D' \cap \mathrm{Fil}^m(D)) \cong \mathrm{Fil}^{m-1}(D) \oplus t^{\mathbf{k}}\mathcal{R}_{K,E}(\delta_m)$$

which is the image of $\mathrm{Fil}^m(D)$ under the pullback map

$$p_{\mathbf{k}} : \mathrm{Ext}^1(\delta_m, \mathrm{Fil}^{m-1}(D)) \to \mathrm{Ext}^1(t^{\mathbf{k}}\delta_m, \mathrm{Fil}^{m-1}(D))$$

which is bijective by Theorem 3.1, contrary to our assumptions (a) and (b). Hence, $m = 1$. Since $D' \cap \mathrm{Fil}^1(D) = D' \cap \mathcal{R}_{K,E}(\delta_1)$ is nonzero and saturated in $\mathcal{R}_{K,E}(\delta_1)$, we conclude that $D' \cap \mathrm{Fil}^1(D) = \mathcal{R}_{K,E}(\delta_1) = \mathrm{Fil}^1(D)$.

In particular, $\mathrm{Fil}^1(D)$ is the unique saturated (φ, Γ_K)-submodule of rank 1 of D. If $n - i > 1$, we pass to the quotient by $\mathrm{Fil}^1(D)$ and claim that every saturated (φ, Γ_K)-submodule of $D/\mathrm{Fil}^1(D)$ contains $\mathrm{Fil}^2(D)/\mathrm{Fil}^1(D)$. By (a) and (b), $D/\mathrm{Fil}^1(D)$ with its induced filtration is non-split and satisfies the analogous (a) and (b), so the argument above works. We can iterate this argument until $n - i$, hence the conclusion. □

Corollary 3.5. *Let $(D, \mathrm{Fil}^\bullet(D))$ be strongly non-split of parameters $\delta \in \mathcal{T}_\circ^n(E)$ over $\mathcal{R}_{K,E}$. Then, any saturated (φ, Γ_K)-submodule D' of D satisfies that $D' = \mathrm{Fil}^i(D)$ for some $0 \leq i \leq n$. In particular, the given triangulation $\mathrm{Fil}^\bullet(D)$ is the unique triangulation of D.*

Proof. This follows from Proposition 3.4 by taking $i = 1$. □

Remark 3.6. One cannot expect the uniqueness result for (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D))$ that is non-split with parameters in $\mathcal{T}_\circ^n(E)$ without the strongly non-split assumption. Indeed, for regular characters $(\delta_1, \delta_2, \delta_3) \in \mathcal{T}_{\mathrm{reg}}^3(E) \cap \mathcal{T}_\circ^3(E)$, we have an exact sequence

$$0 \to \mathrm{Ext}^1(\delta_3, \delta_1) = E \to \mathrm{Ext}^1(\delta_3, (\delta_1 - \delta_2)) = E^2 \to \mathrm{Ext}^1(\delta_3, \delta_2) = E \to 0$$

where $(\delta_1 - \delta_2)$ denotes the non-split extension of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$. The image of any nonzero class of $\mathrm{Ext}^1(\delta_3, \delta_1)$ in $\mathrm{Ext}^1(\delta_3, (\delta_1 - \delta_2))$ is a trianguline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D), (\delta_1, \delta_2, \delta_3))$ that is non-split in the sense of (2.0.4), but $D/\mathrm{Fil}^1(D)$ is a split extension of δ_2 by δ_3. So, D has another triangulation whose parameter is $(\delta_1, \delta_3, \delta_2)$.

3.2. **Crystabelline case.** We discuss triangulations on a class of trianguline (φ, Γ_K)-modules that is closer to p-adic Hodge theory. Recall that we have Fontaine's functors $D_{\mathrm{cris}}, D_{\mathrm{dR}}$ defined for (φ, Γ_K)-modules as well, cf. [HS16, Definition 2.5] or [Nak13, Definition 2.3].

Definition 3.7. A (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ is **crystabelline** if there exists a finite abelian extension L/K such that $\mathcal{R}_{L,E} \otimes_{\mathcal{R}_{K,E}} D$ is crystalline, i.e., the $(L_0 \otimes_{\mathbb{Q}_p} E)$-rank of

$$D^L_{\mathrm{cris}}(D) := (\mathcal{R}_{L,E} \otimes_{\mathcal{R}_{K,E}} D)[1/t]^{\Gamma_L}$$

is equal to $\mathrm{rank}_{\mathcal{R}_{K,E}}(D)$.

As we describe below, one can understand this class of (φ, Γ_K)-modules in terms of simpler semilinear algebra and descent data.

Definition 3.8 ([Nak13, Definition 2.4])**.** Let L be a finite Galois extension of K with Galois group $G(L/K)$. We say that D is an **E-filtered $(\varphi, G(L/K))$-module over K** if

(i) D is a finite free $(L_0 \otimes_{\mathbb{Q}_p} E)$-module with a Frobenius semilinear operator $\varphi : D \xrightarrow{\simeq} D$, and a semilinear action by $G(L/K)$ that commutes with φ.

(ii) $D_L := (L \otimes_{L_0} D)$ has a separated and exhaustive descending filtration $(\mathrm{Fil}^i(D_L))_{i \in \mathbb{Z}}$ by $G(L/K)$-stable $(L \otimes_{\mathbb{Q}_p} E)$-submodules $\mathrm{Fil}^i(D_L)$.

An $(L_0 \otimes_{\mathbb{Q}_p} E)$-module D satisfying (i) will be called a **$(\varphi, G(L/K))$-module over K**.

Remark 3.9. (i) For a $(\varphi, G(L/K))$-module D as in Definition 3.8, we have

$$L \otimes_K (D_L)^{G(L/K)} \xrightarrow{\sim} D_L$$

by Galois descent, and we write $D_K := (D_L)^{G(L/K)}$. Then, a filtration by $G(L/K)$-stable $L \otimes_{\mathbb{Q}_p} E$-submodules on D_L gives rise by taking invariants $(-)^{G(L/K)}$ to a filtration by $K \otimes_{\mathbb{Q}_p} E$-submodules on D_K. Conversely, a filtration by $K \otimes_{\mathbb{Q}_p} E$-submodules on D_K gives rise by base change $L \otimes_K (-)$ to a filtration by $G(L/K)$-stable $L \otimes_{\mathbb{Q}_p} E$-submodules on D_L. Via (2.0.1), $D_K \cong \prod_{\sigma \in \Sigma_K} D_{K,\sigma}$ decomposes into a product of E-vector spaces $D_{K,\sigma}$, and any filtration $\mathrm{Fil}^r D_K = \prod_{\sigma \in \Sigma_K} \mathrm{Fil}^i D_{\mathrm{dR},\sigma}$ also decomposes into a product of filtrations by E-vector spaces on each σ-component $D_{K,\sigma}$.

(ii) For a crystabelline (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ that becomes crystalline over L, $D^L_{\mathrm{cris}}(D)$ has the structure of an E-filtered $(\varphi, G(L/K))$-module whose $G(L/K)$-stable filtration on $L \otimes_{L_0} D^L_{\mathrm{cris}}(D)$ is induced by the comparison isomorphism

$$L \otimes_{L_0} D^L_{\mathrm{cris}}(D) \xrightarrow{\simeq} L \otimes_K D_{\mathrm{dR}}(D),$$

where $(L \otimes_{L_0} D^L_{\mathrm{cris}}(D))^{G(L/K)} \cong D_{\mathrm{dR}}(D)$ has the Hodge filtration $\{\mathrm{Fil}^i(D_{\mathrm{dR}}(D))\}_{i \in \mathbb{Z}}$ by $K \otimes_{\mathbb{Q}_p} E$-submodules, cf. [HS16, Definition 2.5]. Via (2.0.1), we have

$$D_{\mathrm{dR}}(D) = \prod_{\sigma \in \Sigma_K} D_{\mathrm{dR}}(D)_\sigma \quad \text{and} \quad \mathrm{Fil}^i(D_{\mathrm{dR}}(D)) = \prod_{\sigma \in \Sigma_K} \mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma),$$

such that the jumps in the filtration $\prod_{\sigma \in \Sigma_K} \mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma)$ by E-vector spaces are given by $(-1) \cdot (\sigma\text{-Sen weights of } D)$, cf. [KPX14, Definition 6.2.5].

Theorem 3.10. *Let L be a finite Galois extension of K.*

(i) The functor $D \mapsto D^L_{\mathrm{cris}}(D)$ induces a $\otimes$-equivalence of categories from (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$ that become crystalline over L to E-filtered $(\varphi, G(L/K))$-modules over K. The saturated submodules of D correspond to $(\varphi, G(L/K))$-submodules of $D^L_{\mathrm{cris}}(D)$ with their filtrations induced by the Hodge filtration on $D^L_{\mathrm{cris}}(D)$.

(ii) *Let L_0 be the maximal unramified subfield of L, and assume that E contains all $\mathbb{Q}_p$-algebra embeddings of L_0 into $\overline{\mathbb{Q}_p}$. For any free $(L_0 \otimes_{\mathbb{Q}_p} E)$-module D of rank n, we canonically have $D = \prod_{\sigma: L_0 \hookrightarrow E} D_\sigma$, where each D_σ is an E-vector space of rank n. Fix an embedding $\sigma_0 : L_0 \hookrightarrow E$. There is an equivalence of categories from the category of $(\varphi, G(L/K))$-modules D over K to the category of representations (r, V) of the Weil group W_K of K on finite dimensional E-vector spaces such that r is unramified when restricted to the Weil group W_L of L. On the underlying modules, this equivalence is given by $D \mapsto V := D_{\sigma_0}$. The action $r(w)$, for $w \in W_K$, on V is the σ_0-component of the $(L_0 \otimes_{\mathbb{Q}_p} E)$-linear map $r(w) := \overline{w} \circ \varphi^{-\alpha(w)} : D \to D$, where $\alpha(w) \in f\mathbb{Z}$ is the unique integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is given by $x \mapsto x^{p^{\alpha(w)}}$, and $\overline{w} \in \mathrm{Gal}(L/K)$ is the restriction of w to L.*

Proof. (i) is [Nak13, Theorem 2.5] and [Ber17, Proposition 3.3(a)]. Also see [Ber08b].
(ii) is the special case of [BS07, Proposition 4.1] when the monodromy operator $N = 0$. □

Definition 3.11. (i) A crystabelline (φ, Γ_K)-module D is **generic** if the W_K-representation associated to the $(\varphi, G(L/K))$-module $D^L_{\mathrm{cris}}(D)$ is a direct sum of characters $\phi_1, \dots, \phi_n$ of W_K that are trivial on I_L such that if $\phi_1, \dots, \phi_n$ are viewed as smooth characters of $K^\times$ via the local reciprocity map, then $\phi_i/\phi_j \notin \{1, |\cdot|_K^{-1}, |\cdot|_K\}$ for all $i \neq j$, cf. [Din25, §2.1].

(ii) For a crystabelline generic (φ, Γ_K)-module D as in (i), a **refinement** of D is an ordering of the n characters $\phi_1, \dots, \phi_n$ appearing in the associated Weil representation. If we fix an ordering $\phi_1, \dots, \phi_n$, then all the refinements are indexed by elements of S_n, where $w \in S_n$ corresponds to the ordering $\phi_{w(1)}, \dots, \phi_{w(n)}$.

(iii) To each refinement $w \in S_n$ of D as in (ii), we associate a triangulation $\mathrm{Fil}^\bullet_w(D)$ on D by requiring that $\mathrm{Fil}^i_w(D)$ corresponds to the $(\varphi, G(L/K))$-submodule of $D^L_{\mathrm{cris}}(D)$, equipped with the induced filtration, whose associated Weil representation is the subrepresentation $\bigoplus_{1 \le j \le i} \phi_{w(j)}$ of $\bigoplus_{j=1}^n \phi_j$ under the two equivalences recalled in Theorem 3.10, for all i.

Remark 3.12. Following Definition 3.11(iii) and Remark 3.9, under the identifications $(L \otimes_{L_0} D^L_{\mathrm{cris}}(D))^{G(L/K)} = D_{\mathrm{dR}}(D)$ and $D_{\mathrm{dR}}(D) = \prod_{\sigma \in \Sigma_K} D_{\mathrm{dR}}(D)_\sigma$, we recall the relationship between refinements of $D^L_{\mathrm{cris}}(D)$ and the jumps in the induced Hodge filtration $\mathrm{Fil}^\bullet(D_{\mathrm{dR}}(\mathrm{Fil}^i_w(D))_\sigma)$ for $1 \le i \le n$ and $\sigma \in \Sigma_K$. Let $\mathrm{pr}_\sigma : D_{\mathrm{dR}}(D) \to D_{\mathrm{dR}}(D)_\sigma$ be the projection onto the σ-component. For each factor ϕ_j of the W_K-representation $\bigoplus_{j=1}^n \phi_j$, let M_j be the corresponding $(\varphi, G(L/K))$-submodule of $D^L_{\mathrm{cris}}(D)$ of rank 1. Via the construction $(\mathrm{pr}_\sigma) \circ (L \otimes_{L_0} (-))^{G(L/K)}$, M_j gives rise to a 1-dimensional E-subspace of $D_{\mathrm{dR}}(D)_\sigma$, and we choose a basis $e_{j,\sigma}$. For each refinement $w \in S_n$, let $\mathcal{F}^w_\bullet := (0 \subset \mathcal{F}^w_1 \subset \cdots \subset \mathcal{F}^w_n = D^L_{\mathrm{cris}}(D))$ be the complete $(\varphi, G(L/K))$-stable flag on $D^L_{\mathrm{cris}}(D)$ corresponding to the triangulation $\mathrm{Fil}^\bullet_w(D)$. Then, $\mathcal{F}^w_i = \oplus_{j=1}^i M_{w(j)}$, and via the construction $(\mathrm{pr}_\sigma) \circ (L \otimes_{L_0} (-))^{G(L/K)}$, $\mathcal{F}^w_i$ gives rise to an E-subspace $\mathcal{F}^w_{\bullet,\sigma} \subset D_{\mathrm{dR}}(D)_\sigma$ of dimension i with a basis $(e_{w(1),\sigma}, \dots, e_{w(i),\sigma})$. We also have the flag $\mathrm{Fil}^\bullet(D_{\mathrm{dR}}(D)_\sigma)$ coming from the Hodge filtration. The "relative position" of $\mathrm{Fil}^\bullet(D_{\mathrm{dR}}(D)_\sigma)$ and $\mathcal{F}^w_{\bullet,\sigma}$ gives us an element $\mathfrak{w}_\sigma \in S_n$, which is determined by the condition that if the jumps of the σ-Hodge filtration on $\mathcal{F}^w_{n,\sigma} = D_{\mathrm{dR}}(D)_\sigma$ are given by $\{-h_{1,\sigma} < \cdots < -h_{n,\sigma}\}$, then the induced σ-Hodge filtration on $\mathcal{F}^w_i$ has jumps at $\{-h_{\mathfrak{w}_\sigma^{-1}(j),\sigma} | 1 \le j \le i\}$. Thus, the ordering of σ-Sen weights induced by the triangulation $\mathrm{Fil}^\bullet_w(D)$ is $\{h_{\mathfrak{w}_\sigma^{-1}(1),\sigma}, \dots, h_{\mathfrak{w}_\sigma^{-1}(n),\sigma}\}$. We say that the flags $\mathrm{Fil}^\bullet(D_{\mathrm{dR}}(D))$ and $\mathcal{F}^w_\bullet$ are *of relative position* $\mathfrak{w} = (\mathfrak{w}_\sigma)_\sigma \in S_n^{\Sigma_K}$. Then, D is non-critical if and only if the two flags are of relative position $\mathrm{id} \in S_n^{\Sigma_K}$.

Definition 3.13. Let $\mathbf{h} = (h_{1,\sigma} \geq \cdots \geq h_{n,\sigma})_\sigma \in (\mathbb{Z}^n)^{\Sigma_K}$ be the Sen weights of a crystabelline generic (φ, Γ_K)-module D of rank n.

(i) We say $\mathbf{h}$ is **regular** if $h_{1,\sigma} > \cdots > h_{n,\sigma}$ for each $\sigma \in \Sigma_K$.

(ii) A refinement of the crystabelline generic (φ, Γ_K)-module D

$$w(\phi) = (\phi_{w(1)}, \ldots, \phi_{w(n)}), \text{ for } w \in S_n,$$

is **non-critical** if the σ-Sen weights of $\mathrm{Fil}^i_w(D)$ are $(h_{1,\sigma} \geq \cdots \geq h_{i,\sigma})$ for all $1 \leq i \leq n$ and $\sigma \in \Sigma_K$. We call a refinement of D **critical** if it is not non-critical. Then, D is called **non-critical** if all its refinements are non-critical, and is called **critical** if at least one of its refinements is critical.

Denote by $\Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ the class of all crystabelline non-critical (φ, Γ_K)-modules of rank n over $\mathcal{R}_{K,E}$ with associated W_K-representation $\phi = \bigoplus_{i=1}^n \phi_i$ and Sen weights $\mathbf{h}$.

Proposition 3.14. *Let D be a crystabelline generic (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,E}$ that is crystalline over a finite abelian extension L/K, with Weil representation $\phi = \bigoplus_{i=1}^n \phi_i$ and Sen weights $\mathbf{h} = (h_{1,\sigma} \geq \cdots \geq h_{n,\sigma})_{\sigma \in \Sigma_K}$. Then, D is trianguline and has $n!$ triangulations*

$$\mathrm{Fil}^\bullet_w(D): \quad 0 \subsetneq \mathrm{Fil}^1_w(D) \subsetneq \cdots \subsetneq \mathrm{Fil}^n_w(D) = D,$$

which are indexed by refinements $w \in S_n$, and are of parameters $\delta_w = (\delta_{w,1}, \ldots, \delta_{w,n})$ with

$$\delta_{w,i} = \left(\prod_{\sigma \in \Sigma_K} x_\sigma^{k^w_{i,\sigma}} \right) \phi_{w(i)}$$

for some integer $k^w_{i,\sigma} \in \mathbb{Z}$ such that $\{h_{1,\sigma}, \ldots, h_{n,\sigma}\} = \{k^w_{1,\sigma}, \ldots, k^w_{n,\sigma}\}$ as sets, for all σ. Moreover, D is non-critical if and only if $k^w_{i,\sigma} = h_{i,\sigma}$, for all $1 \leq i \leq n$, $w \in S_n$, and $\sigma \in \Sigma_K$.

Proof. The last assertion would follow from the previous assertions and the definition of non-critical. The genericity of D implies that there are $n!$ complete flags consisting of subobjects in the Weil representation $\bigoplus_{i=1}^n \phi_i$ attached to D. By the equivalences in Theorem 3.10, the flags give rise to $n!$ distinct triangulations on D, indexed by permutations $w \in S_n$ of the set $\{\phi_1, \ldots, \phi_n\}$. It remains to show that the triangulation parameters are of the forms given in the statement. Since the property of being crystabelline is stable under taking subquotients, by considering $\mathrm{Fil}^i_w(D)/\mathrm{Fil}^{i-1}_w(D)$ we are reduced to showing the following claim on crystabelline (φ, Γ_K)-modules of rank 1: if $\mathcal{R}_{K,E}(\delta)$ is a crystabelline (φ, Γ_K)-module of rank 1 that becomes crystalline over L, then $\delta = \widetilde{\delta} \prod_{\sigma \in \Sigma_K} (x_\sigma^{k_\sigma})$ for a smooth character $\widetilde{\delta} : K^\times \to E^\times$ and $k_\sigma \in \mathbb{Z}$, and the W_K-representation ϕ attached to the $(\varphi, G(L/K))$-module $D^L_{\mathrm{cris}}(\mathcal{R}_{K,E}(\delta))$ is $\widetilde{\delta} \circ \mathrm{rec}_K^{-1}$.

By construction, the W_K-representation ϕ on E-vector spaces is independent of the choice of L over which $R_{K,E}(\delta)$ becomes crystalline. Observe that any crystabelline (φ, Γ_K)-module M becomes crystalline over a totally ramified abelian extension L/K. For this observation, we can prove it for the B-pair $W = (W_e, W^+_{\mathrm{dR}})$ associated to M in [Nak09, §1.3], since $D^L_{\mathrm{cris}}(M) = D^L_{\mathrm{cris}}(W)$ by [Ber08a, Proposition 2.3.4]. By local class field theory, there are $m \in \mathbb{Z}_{\geq 1}$ and a finite unramified extension L'/K such that $L \subset K_m.L'$, where K_m is the m-th Lubin-Tate extension of K. We have the short exact sequence of Galois groups

$$1 \to G_{K_m.L'} \to G_{K_m} \to \mathrm{Gal}(K_m.L'/K_m) \to 1.$$

Since $L \subset K_m.L'$, it follows that $G_{K_m.L'} \subset G_L$, and M becomes crystalline over $K_m.L'$. Then,

$$D^{K_m.L'}_{\mathrm{cris}}(W) := (B_{\max} \otimes_{B_e} W_e)^{G_{K_m.L'}} \cong D^{K_m.L'}_{\mathrm{cris}}(M)$$

is a semilinear $\mathrm{Gal}(K_m.L'/K_m)$-module over $(K_m.L')_0 = L'_0$. By Galois descent, we have

$$D_{\mathrm{cris}}^{K_m}(W)\otimes_{(K_m)_0}(K_m.L')_0 = (B_{\max}\otimes_{B_e} W_e)^{G_{K_m}}\otimes_{K_0} L'_0 \xrightarrow{\simeq} (B_{\max}\otimes_{B_e} W_e)^{G_{K_m.L'}} = D_{\mathrm{cris}}^{K_m.L'}(W),$$

which shows that D becomes crystalline over K_m. Hence, from now on we may take $L = K_m$ for $m \gg 0$, which is totally ramified over K, in our proof of the claim.

Since $\mathcal{R}_{K,E}(\delta)$ is crystabelline, $\delta = \widetilde{\delta}\prod_{\sigma\in\Sigma_K}(x_\sigma^{k_\sigma})$ for some smooth character $\widetilde{\delta} : K^\times \to E^\times$ and integers $k_\sigma \in \mathbb{Z}$ by [Nak09, Lemma 4.1]. We need to show that the Weil representation ϕ attached to $D_{\mathrm{cris}}^{K_m}(R_{K,E}(\delta))$ is $\widetilde{\delta}\circ \mathrm{rec}_K^{-1}$. Let $G_m := \mathrm{Gal}(K_m/K)$ and write $\widetilde{\delta} = \widetilde{\delta}^{\mathrm{unr}}.\widetilde{\delta}^{\mathrm{wt}}$ so that $\widetilde{\delta}^{\mathrm{unr}}(\pi_K) = \widetilde{\delta}(\pi_K)$ and $\widetilde{\delta}^{\mathrm{wt}}|_{\mathcal{O}_K^\times} = \widetilde{\delta}|_{\mathcal{O}_K^\times}$. We firstly compute the (φ, G_m)-module $D_{\mathrm{cris}}^{K_m}(R_{K,E}(\delta))$ by Theorem 3.10(i), and then compute its Weil representation by Theorem 3.10(ii). We have

$$D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\delta)) = D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{wt}}))\otimes_{K_0\otimes_{\mathbb{Q}_p}E} D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{unr}}\prod_{\sigma\in\Sigma_K} x_\sigma^{k_\sigma})).$$

By $(K_m)_0 = K_0$ and applying to $\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{unr}}\prod_{\sigma\in\Sigma_K} x_\sigma^{k_\sigma})$ [KPX14, Example 6.2.6(3)], we have

$$D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{unr}}\prod_{\sigma\in\Sigma_K} x_\sigma^{k_\sigma})) = (K_m)_0\otimes_{K_0} D_{\mathrm{cris}}(\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{unr}}\prod_{\sigma\in\Sigma_K} x_\sigma^{k_\sigma})),$$

where $G(K_m/K)$ acts trivially, and as a φ-module, it is the free $(K_0\otimes_{\mathbb{Q}_p}E)$-module $D_{f,\widetilde{\delta}(\pi_K)}$ of rank 1 equipped with a Frobenius-linear endomorphism φ such that $\varphi^f = 1\otimes\widetilde{\delta}(\pi_K)$, which is unique up to isomorphism by [KPX14, Lemma 6.2.3]. As for $D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\widetilde{\delta}^{\mathrm{wt}}))$, since $\widetilde{\delta}^{\mathrm{wt}}(\pi_K) = 1$, it extends by continuity to a continuous character of G_K taking the value 1 at any geometric Frobenius. Since $\widetilde{\delta}$ is smooth, we may also assume that $\widetilde{\delta}^{\mathrm{wt}}|_{1+\pi_K^m\mathcal{O}_K} = 1$, i.e., the restriction of $\widetilde{\delta}^{\mathrm{wt}}$ to G_{K_m} is the trivial character. Then, $\widetilde{\delta}^{\mathrm{wt}}$ descends to a character of G_m, and

$$D_{\mathrm{cris}}^{K_m}(\widetilde{\delta}^{\mathrm{wt}}) = \left(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p} E(\widetilde{\delta}^{\mathrm{wt}})\right)^{G_{K_m}} = K_0\otimes_{\mathbb{Q}_p} E(\widetilde{\delta}^{\mathrm{wt}})$$

which is the free $(K_0\otimes_{\mathbb{Q}_p}E)$-module $D_{f,1}$ of rank 1 on which G_m acts by $\widetilde{\delta}^{\mathrm{wt}}$. Thus, we conclude that $D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\delta))$ is the φ-module $D_{f,\widetilde{\delta}(\pi_K)}$ free of rank 1 over $(K_0\otimes_{\mathbb{Q}_p}E)$, where $G(K_m/K)$ acts by $\widetilde{\delta}^{\mathrm{wt}}$. To show that the Weil representation ϕ attached to $D_{\mathrm{cris}}^{K_m}(\mathcal{R}_{K,E}(\delta))$ is $\widetilde{\delta}\circ\mathrm{rec}_K^{-1}$, we follow the recipe recalled in Theorem 3.10(ii). For $w\in W_K$, let $\alpha(w)\in f\mathbb{Z}$ be the integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}_p}|\mathbb{F}_p)$ is the $\alpha(w)$-th power of the absolute arithmetic Frobenius. Then the W_K-action is given by $\phi(w) = \overline{w}\circ\varphi^{-\alpha(w)} = \widetilde{\delta}^{\mathrm{wt}}(\mathrm{rec}_K^{-1}(w))\cdot\varphi^{-\alpha(w)}$, where $\overline{w}$ denotes the image of w in G_m. Since $\mathrm{rec}_K(\pi_K)$ is a geometric Frobenius element,

$$\phi(\mathrm{rec}_K(\pi_K)) = \widetilde{\delta}^{\mathrm{wt}}(\pi_K)\varphi^{-(-f)} = \widetilde{\delta}^{\mathrm{wt}}(\pi_K)\widetilde{\delta}(\pi_K) = \widetilde{\delta}(\pi_K).$$

For $u\in\mathcal{O}_K^\times = \mathrm{rec}_K^{-1}(I_K^{\mathrm{ab}})$, we have $\alpha(\mathrm{rec}_K(u)) = 0$ and hence

$$\phi(\mathrm{rec}_K(u)) = \widetilde{\delta}^{\mathrm{wt}}(u)\varphi^{-(0)} = \widetilde{\delta}(u).$$

We conclude that $\phi\circ\mathrm{rec}_K = \widetilde{\delta}$, and hence $\phi = \widetilde{\delta}\circ\mathrm{rec}_K^{-1}$, which proves the claim. ∎

Proposition 3.15. *For $D\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ of regular Sen weight $(h_{1,\sigma} > \cdots > h_{n,\sigma})$ for each $\sigma\in\Sigma_K$, D is indecomposable. In particular, all of its triangulations are strongly non-split.*

Proof. Any direct sum decomposition $D = D' \oplus D''$ of D with two (φ, Γ_K)-submodules D', D'' is part of a triangulation $\mathrm{Fil}_w^\bullet(D)$ for some $w \in S_n$, i.e., $D' \cong \mathrm{Fil}_w^i(D)$ for $i = \mathrm{rank}(D')$ and $D'' \cong D/\mathrm{Fil}_w^i(D)$. This is because D is crystabelline generic, and hence by Theorem 3.10 and Definition 3.11(i), its saturated subobjects corresponds to the subobjects of the semisimple Weil representation $\bigoplus_i \phi_i$. As in Remark 2.11(i), the problem is reduced to the rank 2 case. It suffices to show that for any non-critical crystabelline (φ, Γ_K)-module D of rank 2 of regular weights, if we have an extension $0 \to \mathcal{R}_{K,E}(\delta_1) \to D \to \mathcal{R}_{K,E}(\delta_2) \to 0$, then this extension is non-split. This is true since by the non-critical assumption, $\mathrm{wt}_\sigma(\delta_1) > \mathrm{wt}_\sigma(\delta_2)$ for all $\sigma \in \Sigma_K$. If the extension is split, we can write D as an extension $0 \to \mathcal{R}_{K,E}(\delta_2) \to D \to \mathcal{R}_{K,E}(\delta_1) \to 0$, which is a critical triangulation of D. $\square$

Proposition 3.16. *(i) Let D be a generic crystabelline $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_{\mathbb{Q}_p,E}$ of regular Sen weight $\mathbf{h} = (h_1 > \cdots > h_n)$ and Weil representation $\bigoplus_{i=1}^n \phi_i$. If D has a critical refinement w, then $(D, \mathrm{Fil}_w^\bullet(D))$ is not strongly non-split in the sense of (2.0.5).*

(ii) For any nontrivial extension $K/\mathbb{Q}_p$, regular Sen weight $\mathbf{h} = (h_{1,\sigma} > \cdots > h_{n,\sigma})_{\sigma \in \Sigma_K}$, and generic Weil representation $\bigoplus_{i=1}^n \phi_i$, there exists a generic crystabelline (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$ of regular Sen weight $\mathbf{h}$ and Weil representation $\bigoplus_{i=1}^n \phi_i$ that has a critical refinement w such that $(D, \mathrm{Fil}_w^\bullet(D))$ is strongly non-split in the sense of (2.0.5).

Proof. (i) Let D be a generic crystabelline $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_{\mathbb{Q}_p,E}$ of regular Sen weight $\mathbf{h}$ and Weil representation $\bigoplus_{i=1}^n \phi_i$. Suppose $w \in S_n$ is a critical refinement of D, which means that the induced ordering of weights satisfies $k_i^w < k_{i+1}^w$ for some $1 \le i \le n-1$, where $\{k_i^w, k_{i+1}^w\} \subset \{h_1, \dots, h_n\}$, cf. Proposition 3.14. We claim that the extension

$$0 \to \mathcal{R}_{\mathbb{Q}_p,E}(x^{k_i^w}\phi_{w(i)}) \to \mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^{i-1}(D) \to \mathcal{R}_{\mathbb{Q}_p,E}(x^{k_{i+1}^w}\phi_{w(i+1)}) \to 0$$

is split, where the subobject is $\mathrm{Fil}_w^i(D)/\mathrm{Fil}_w^{i-1}(D)$ and the quotient is $\mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^i(D)$. In the notations of Remark 3.12, since the induced jumps are $-\mathrm{wt}(\delta_{w,i}) > -\mathrm{wt}(\delta_{w,i+1})$, the refinement $Ee_{w(i)} \subset Ee_{w(i)} \oplus Ee_{w(i+1)}$ on $D_{\mathrm{dR}}(\mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^i(D))$ coincides with the Hodge filtration, which forces the other refinement $Ee_{w(i+1)} \subset Ee_{w(i)} \oplus Ee_{w(i+1)}$ to be of relative position id with the Hodge filtration. This other refinement gives a triangulation

$$0 \to \mathcal{R}_{\mathbb{Q}_p,E}(x^{k_{i+1}^w}\phi_{w(i+1)}) \to \mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^{i-1}(D)) \to \mathcal{R}_{\mathbb{Q}_p,E}(x^{k_i^w}\phi_{w(i)}) \to 0\,.$$

Since D is generic, $\phi_{w(i)} \neq \phi_{w(i+1)}$. So by Lemma 2.7(iii), the intersection of the subobjects $\mathcal{R}_{\mathbb{Q}_p,E}(x^{k_i^w}\phi_{w(i)})$ and $\mathcal{R}_{\mathbb{Q}_p,E}(x^{k_{i+1}^w}\phi_{w(i+1)})$ is zero. The subobject $\mathcal{R}_{\mathbb{Q}_p,E}(x^{k_{i+1}^w}\phi_{w(i+1)})$ reduces isomorphically to the quotient $\mathcal{R}_{\mathbb{Q}_p,E}(x^{k_{i+1}^w}\phi_{w(i+1)})$ of $\mathrm{Fil}_w^{i+1}(D)/\mathrm{Fil}_w^i(D))$ in the first extension above, making it a split extension.

(ii) The generic Weil representation $\bigoplus_{j=1}^n \phi_j$ determines a unfiltered $(\varphi, G(L/K))$-module M. To get a crystabelline (φ, Γ_K)-module D of Sen weight $\mathbf{h}$ with $D_{\mathrm{cris}}^L(D) = M$, we need to give filtrations $\mathrm{Fil}^i(D_{\mathrm{dR}}(D)_\sigma)$ on $D_{\mathrm{dR}}(D)_\sigma := (L \otimes_{L_0} M_j)_\sigma^{G(L/K)}$ whose jumps occur at $\{-h_{1,\sigma} < \cdots < -h_{n,\sigma}\}$ for each $\sigma \in \Sigma_K$. Since the property of being strongly non-split for $(D, \mathrm{Fil}_w^\bullet(D))$ can be checked at subquotients of rank 2, it suffices to construct an example D of rank 2 with Sen weights $(h_{1,\sigma} > h_{2,\sigma})_{\sigma \in \Sigma_K}$ and generic Weil representation $\phi_1 \oplus \phi_2$, which has a critical refinement $w \in S_2$ such that $\mathrm{Fil}_w^\bullet(D)$ is non-split. As in Remark 3.12, for the refinement (ϕ_1, ϕ_2), we choose a basis $(e_{1,\sigma}, e_{2,\sigma})_{\sigma \in \Sigma_K}$ for $D_{\mathrm{dR}}(D) = \prod_\sigma D_{\mathrm{dR}}(D)_\sigma$. Since $K \neq \mathbb{Q}_p$, there are at least two distinct embeddings $\sigma_1, \sigma_2 \in \Sigma_K$. Define

$$D_{\mathrm{dR}}(D)_{\sigma_1} \supsetneq \mathrm{Fil}^{-h_{1,\sigma_1}+1}(D_{\mathrm{dR}}(D)_{\sigma_1}) := E(e_{1,\sigma_1} + e_{2,\sigma_1}) \supsetneq \mathrm{Fil}^{-h_{2,\sigma_1}+1}(D_{\mathrm{dR}}(D)_{\sigma_1}) = 0$$

$$D_{\mathrm{dR}}(D)_{\sigma_2} \supsetneq \quad \mathrm{Fil}^{-h_{1,\sigma_2}+1}(D_{\mathrm{dR}}(D)_{\sigma_2}) := E(e_{1,\sigma_2}) \supsetneq \mathrm{Fil}^{-h_{2,\sigma_2}+1}(D_{\mathrm{dR}}(D)_{\sigma_2}) = 0$$

and define the τ-Hodge filtrations for $\tau \in \Sigma_K \setminus \{\sigma_1, \sigma_2\}$ arbitrarily. This gives a (φ, Γ_K)-module D. By Remark 3.12, $\mathrm{Fil}^\bullet_{\mathrm{id}}(D)$ induces the ordering $(h_{1,\sigma_1} > h_{2,\sigma_1})$ on the σ_1-Sen weights and the ordering $(h_{2,\sigma_2} < h_{1,\sigma_2})$ on the σ_2-Sen weights, but for $w_0 = (1\ 2) \in S_2$, $\mathrm{Fil}^\bullet_{w_0}(D)$ induces the ordering $(h_{1,\sigma_i} > h_{2,\sigma_i})$ on the σ_i-Sen weights for both $i = 1, 2$. This implies that $\mathrm{Fil}^\bullet_{\mathrm{id}}(D)$ is non-split, for in the split case, $\mathrm{Fil}^\bullet_{w_0}(D)$ would induce the orderings $(h_{2,\sigma_1} < h_{1,\sigma_1})$ on the σ_1-Sen weights and $(h_{1,\sigma_2} > h_{2,\sigma_2})$ on the σ_2-Sen weights. □

Remark 3.17. For critical generic crystalline (φ, Γ_K)-module D, it may be strongly non-split for some non-critical refinement $\mathrm{Fil}^\bullet(D)$, and it may have no strongly non-split non-critical refinement at all. Hence, "strongly non-split" depends on the choice of triangulation.

3.3. **More general cases.**

Remark 3.18. We end this section with a heuristic procedure for locating possible triangulations of a fixed (φ, Γ_K)-module D over $\mathcal{R}_{K,E}$, where E is a finite extension of $\mathbb{Q}_p$ containing the Galois closure K^{norm} of K. This discussion will not be used later. As the first step, consider the locus X_1 in $\mathcal{T}_E(E)$ where the character $\delta_1 = \delta$ makes the space

$$\left\{ f_1 \in H^0(D^\vee(\delta_1)) \middle| \forall \sigma \in \Sigma_K, \overline{f_1} \neq 0 \in H^0(D^\vee(\delta_1)/t_\sigma) \right\}$$

nonempty, where $\overline{f_1}$ denotes the image of f under the mod-t_σ reduction map

$$\mathrm{Hom}_{\varphi,\Gamma_K}(D, \mathcal{R}_{K,E}(\delta)) \to \mathrm{Hom}_{\varphi,\Gamma_K}(D/t_\sigma, \mathcal{R}_{K,E}(\delta)/t_\sigma).$$

By the classification of (φ, Γ_K)-submodules of $\mathcal{R}_{K,E}(\delta_1)$ in Lemma 2.7(iii), any f_1 in the space above corresponds to a map $f_1 : D \to \mathcal{R}_{K,E}(\delta_1)$ of (φ, Γ_K)-modules that is surjective, and vice versa. To the data of (δ_1, f_1), we associate a (φ, Γ_K)-submodule of D of rank $n-1$ as

$$0 \to D_{n-1} := \ker(f_1) \to D \xrightarrow{f_1} \mathcal{R}_{K,E}(\delta_1) \to 0.$$

The next step is to consider, for all (f_1, δ_1) found in Step 1, the locus X_2 in $\mathcal{T}_E(E)$ where the character $\delta_2 = \delta$ makes the space

$$\left\{ f_2 \in H^0(D_{n-1}^\vee(\delta_2)) \middle| \forall \sigma \in \Sigma_K, \overline{f_2} \neq 0 \in H^0(D_{n-1}^\vee(\delta_2)/t_\sigma) \right\}$$

nonempty. Any f_2 from this space corresponds to a surjective map from D_{n-1} to $\mathcal{R}_{K,E}(\delta_2)$ of (φ, Γ_K)-modules, whose kernel (we denote by D_{n-2}) satisfies

$$0 \to D_{n-2} := \ker(f_2) \to D_{n-1} \xrightarrow{f_2} \mathcal{R}_{K,E}(\delta_2) \to 0.$$

Continuing this way, we get data $(\delta_1, f_1, \delta_2, f_2, \ldots, \delta_{n-1}, f_{n-1}, \delta_n)$, where for $D_{n-i} := \ker(f_i)$ with $D_n = D$, f_i are chosen from

$$\left\{ f_i \in H^0(D_{n-i+1}^\vee(\delta_i)) \middle| \forall \sigma \in \Sigma_K, \overline{f_i} \neq 0 \in H^0(D_{n-i+1}^\vee(\delta_i)/t_\sigma) \right\},$$

from which we know there is a triangulation $\mathrm{Fil}^\bullet(D) = D_\bullet$ on D with parameters $(\delta_n, \ldots, \delta_1)$.

4. Pullback operations

Consider the stack $\mathfrak{X}_n = \mathfrak{X}_{\mathrm{GL}_n}$ of G_K-equivariant vector bundles of rank n on the Fargues-Fontaine curve $X_{\overline{K}}$, the stack $\mathfrak{X}_P$ of G_K-equivariant P-bundles on $X_{\overline{K}}$ for a standard parabolic subgroup $P \subset \mathrm{GL}_n$ containing the upper triangular Borel subgroup $B \subset \mathrm{GL}_n$ with Levi quotient $M \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, and the stack $\mathfrak{X}_M$ of G_K-equivariant M-bundles on $X_{\overline{K}}$, all over the category of rigid E-analytic spaces Rig_E equipped with the Tate-fpqc topology, cf. [EGH, §5.1, §5.3]. Note that $\mathfrak{X}_M \cong \mathfrak{X}_{n_1} \times \cdots \times \mathfrak{X}_{n_r}$. When $P = B$, the Levi quotient M is the diagonal torus, which we denote by $T \cong (\mathrm{GL}_1)^n$ so that $\mathfrak{X}_T = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_1$.

By [EGH, 5.1.1 and 5.1.5], a point of $\mathfrak{X}_n(\mathrm{Sp}(A))$ is a (φ, Γ_K)-module D of rank n over $\mathcal{R}_{K,A}$, and a point of $\mathfrak{X}_P(\mathrm{Sp}(A))$ is a pair $(D, \mathrm{Fil}^\bullet(D))$ consisting of a (φ, Γ_K)-module D of rank n over $\mathcal{R}_{K,A}$ together with a filtration

$$0 = \mathrm{Fil}^0(D) \subset \mathrm{Fil}^1(D) \subset \cdots \subset \mathrm{Fil}^r(D) = D$$

by (φ, Γ_K)-submodules with the subquotients $\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)$ being locally free over $\mathcal{R}_{K,A}$ of rank n_i for all $1 \leq i \leq r$, in which case we say that $\mathrm{Fil}^\bullet(D)$ is a P**-parabolization** on D. In general, a (φ, Γ_K)-module D of rank n is called **paraboline** if it admits a P-parabolization for some parabolic subgroup P of GL_n containing B.

For P_i the parabolic subgroup containing B of Levi quotient $M_i \cong \mathrm{GL}_{n-i} \times \mathrm{GL}_i$, we may consider the pullback map $p_{i,\sigma} : \mathfrak{X}_{P_i} \to \mathfrak{X}_{P_i}$, sending a P_i-bundle

$$(0 \to D_{n-i} \to D \to D^i := D/D_{n-i} \to 0),$$

where D_{n-i} is a saturated (φ, Γ_K)-submodule of D of rank $n-i$, to the P_i-subbundle

$$(0 \to D_{n-i} \to p_{i,\sigma}(D) \to t_\sigma D^i \to 0).$$

This makes sense on $\mathfrak{X}_Q$ for any parabolic Q such that $B \subset Q \subset P_i$, where $p_{i,\sigma}$ acts similarly.

Remark 4.1. On $\mathfrak{X}_B$, consider $p_{i,\sigma} : \mathfrak{X}_B \to \mathfrak{X}_B$. Over the sublocus of $\mathcal{T}^n$ where $\mathrm{wt}_\sigma(\delta_j/\delta_k) \neq 1$ for all $1 \leq j \leq n-i < k \leq n$ with fixed i, and for a non-split pair $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_n(B)$, it follows from Theorem 3.1 that from $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ together with its induced triangulation, we can recover D together with its triangulation.

Lemma 4.2. *For any $1 \leq i < j \leq n$ and $\sigma, \tau \in \Sigma_K$ (with possibly $\sigma = \tau$), we have that*

$$p_{i,\sigma} \circ p_{j,\tau} = p_{j,\tau} \circ p_{i,\sigma}$$

as maps from $\mathfrak{X}_Q$ to $\mathfrak{X}_Q$, for any parabolic Q such that $B \subset Q \subset P_i \cap P_j$.

Proof. We may assume $Q = P_i \cap P_j$. Then, a Q-bundle is $D \in \mathfrak{X}_n(A)$ together with a filtration

$$0 \subset D_{n-j} := \mathrm{Fil}^{n-j}(D) \subset D_{n-i} := \mathrm{Fil}^{n-i}(D) \subset D_n = D$$

by (φ, Γ_K)-submodules over $\mathcal{R}_{K,A}$ such that $D^\bullet := D/D_{n-\bullet}$ is locally free of rank $\bullet \in \{i, j\}$, and $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D)) = (D_{n-i} - t_\sigma D^i)$, which has the induced Q-filtration

$$0 \subset D_{n-j} \subset D_{n-i} \subset p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D)).$$

From this we see that, because the twist by t_τ commutes with pullback via t_σ,

$$\begin{aligned} p_{j,\tau}(p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))) &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j} - t_\sigma D^i)) \\ &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j}) - t_\tau t_\sigma D^i) \\ &= (D_{n-j} - t_\tau(D_{n-i}/D_{n-j}) - t_\sigma(t_\tau D^i)) \end{aligned}$$

$$= p_{i,\sigma}(p_{j,\tau}(D, \mathrm{Fil}^\bullet(D)))\,,$$

which completes the proof. □

4.1. **Substacks.** We define several substacks of $\mathfrak{X}_n$ in this subsection. We will see in Theorem 4.12(i) that $p_{i,\sigma}$ descend to them.

Definition 4.3. (i) Recall we have the morphisms of stacks

$$\begin{array}{ccccc} & & \mathfrak{X}_P & & \\ & \swarrow^{\beta_P} & & \searrow^{\alpha_P} & \\ \mathfrak{X}_n & & & & \mathfrak{X}_M \end{array}$$

where for a paraboline (φ, Γ_K)-module $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_P(A)$, we have the forgetful map

$$\beta_P(D, \mathrm{Fil}^\bullet(D)) = D,$$

and the map

$$\alpha_P(D, \mathrm{Fil}^\bullet(D)) = (\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D))_i$$

taking the successive quotients of the P-structure $\mathrm{Fil}^\bullet(D)$ on D.

(ii) For $P = B$, we define the **weight maps**

$$\omega_T : \mathfrak{X}_T \to \mathrm{Res}_{K/\mathbb{Q}_p}(\mathbb{A}^{n,\mathrm{rig}})_E \cong (\mathbb{A}^n_E)^{\Sigma_K,\mathrm{rig}}, \quad (\delta_1, \dots, \delta_n) \mapsto (\mathrm{wt}(\delta_1), \dots, \mathrm{wt}(\delta_n))$$

and

$$\omega_B : \mathfrak{X}_B \xrightarrow{\alpha_B} \mathfrak{X}_T \xrightarrow{\omega_T} (\mathbb{A}^n_E)^{\Sigma_K,\mathrm{rig}}, \quad (D, \mathrm{Fil}^\bullet(D)) \mapsto (\mathrm{wt}(\delta_1), \dots, \mathrm{wt}(\delta_n))$$

where δ_i are the unique characters determined by the rank 1 subquotients

$$\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D) \cong \mathcal{R}_{K,A}(\delta_i) \otimes_{\mathcal{O}_{\mathrm{Sp}(A)}} \mathcal{L}_i.$$

With the projection

$$\mathrm{pr}_\sigma : (\mathbb{A}^n_E)^{\Sigma_K,\mathrm{rig}} \to \mathbb{A}^{n,\mathrm{rig}}_E$$

onto the σ-component for $\sigma \in \Sigma_K$, we define the **σ-weight maps** by

$$\omega_{T,\sigma} := \mathrm{pr}_\sigma \circ \omega_T : \mathfrak{X}_T \to \mathbb{A}^{n,\mathrm{rig}}_E, \quad (\delta_1, \dots, \delta_n) \mapsto (\mathrm{wt}_\sigma(\delta_1), \dots, \mathrm{wt}_\sigma(\delta_n))$$

and

$$\omega_{B,\sigma} := \omega_{T,\sigma} \circ \alpha_B : \mathfrak{X}_B \to \mathbb{A}^{n,\mathrm{rig}}_E, \quad (D, \mathrm{Fil}^\bullet(D)) \mapsto (\mathrm{wt}_\sigma(\delta_1), \dots, \mathrm{wt}_\sigma(\delta_n)).$$

Definition 4.4. Let $S \subset \Sigma_K$ be a subset of embeddings of K into E.

(i) For a trianguline (φ, Γ_K)-module D over $\mathcal{R}_{K,A}$, we say D is **S-weight-uniform trianguline** if for every rigid point $z \in \mathrm{Sp}(A)$ and every finite extension $L/k(z)$, the σ-weight map $\omega_{B,\sigma}$ is constant on $\beta_B^{-1}(D_z \otimes_{k(z)} L)$ for every $\sigma \in S$. That is, all triangulations on the fiber $D_z \otimes_{k(z)} L$ induce the same ordering on its σ-Sen weights for all $\sigma \in S$.

(ii) As the S-weight-uniform trianguline condition is defined via geometric fibers, it is stable under base change. Denote by $\mathfrak{X}_B^{S\text{-wu}}$ this substack of $\mathfrak{X}_B$, and by $\mathfrak{X}_n^{S\text{-wu}}$ the sheafification of its image under β_B in $\mathfrak{X}_n$ with respect to the Tate-fpqc topology on Rig_E. We refer to $\mathfrak{X}_B^{S\text{-wu}}$ and $\mathfrak{X}_n^{S\text{-wu}}$ as the **S-weight-uniform substacks**.

(iii) If $S = \Sigma_K$, we simply write $\mathfrak{X}_B^{\mathrm{wu}}$ and $\mathfrak{X}_n^{\mathrm{wu}}$, and call them **weight-uniform substacks**.

Remark 4.5. Sheafification is necessary for descent-theoretic reasons. [EGH, Example 5.3.2] gives a filtered φ-module on $\mathbb{P}^1$ defining a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module of rank 2 on $\mathbb{P}^1$, which has no global triangulation on $\mathbb{P}^1$, but is globally trianguline over the two copies of $\mathbb{A}^1$.

Definition 4.6. (i) Fix $\sigma \in \Sigma_K$ and $i \in \{1, \dots, n\}$.

(a) Let U_i be the Zariski-open subspace of $\mathbb{A}_E^{n,\mathrm{rig}}$ that is complement to the hyperplanes $\{T_j - T_k = 0\}$ for all $1 \le j \le n-i$ and $n-i+1 \le k \le n$, where we denote by $\{T_l | 1 \le l \le n\}$ the standard coordinates on $\mathbb{A}_E^{n,\mathrm{rig}}$.

(b) Let $\mathfrak{X}_n^{\sigma\text{-wu},i}$ denote the sheafification of the image $\beta_B(\mathfrak{X}_B^{\sigma\text{-wu}} \cap \omega_{B,\sigma}^{-1}(U_i))$, which is the substack of $\mathfrak{X}_n^{\sigma\text{-wu}}$ characterized by the property that $D \in \mathfrak{X}_n^{\sigma\text{-wu},i}(A)$ if and only if for any $z \in \mathrm{Sp}(A)$ and for any triangulation on the fiber $D_z \in \mathfrak{X}_n(k(z))$, its first $n-i$ ordered σ-Sen weights are disjoint from its last i ordered σ-Sen weights in $k(z)$.

(c) For $a \le 0 \le b \in \mathbb{Z}$, let $\mathfrak{X}_n^{\sigma\text{-wu},i,[a,b]}$ be the substack of $\mathfrak{X}_n^{\sigma\text{-wu}}$ given by the condition that $D \in \mathfrak{X}_n^{\sigma\text{-wu},i,[a,b]}(A)$ if and only if for any $z \in \mathrm{Sp}(A)$ and for any triangulation on D_z with parameters $(\delta_{1,z}, \dots, \delta_{n,z})$, we have

$$\{\mathrm{wt}_\sigma(\delta_{j,z}) | 1 \le j \le n-i\} \cap \{\mathrm{wt}_\sigma(\delta_{k,z}) + h | n-i+1 \le k \le n, h \in [a,b] \cap \mathbb{Z}\} = \emptyset$$

in $k(z)$. In particular, $\mathfrak{X}_n^{\sigma\text{-wu},i}$ defined in (b) is equal to $\mathfrak{X}_n^{\sigma\text{-wu},i,[0,0]}$.

(ii) Fix $S \subset \Sigma_K$, and for each $\sigma \in S$, choose a subset

$$I_\sigma := \{1 \le i_1 < \cdots < i_{d_\sigma} \le n\}$$

of $\{1, \dots, n\}$ of size d_σ. Set $i_0 = 0$ and $i_{d_\sigma+1} = n$. For $\mathbf{k} = (k_{\sigma,i})_{\sigma \in S, i \in I_\sigma} \in \mathbb{N}^{\sum_{\sigma \in S} d_\sigma}$, let

$$\bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,\left[0,\sum_{r=1}^m k_{\sigma,i_r}\right]} \subset \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}} \subset \bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,[0,k_{\sigma,i_m}]}$$

be the substack of $\mathfrak{X}_n^{S\text{-wu}}$ given by the condition that $D \in \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}}(A)$ if and only if for any $z \in \mathrm{Sp}(A)$ and for any triangulation on D_z with parameters $(\delta_{1,z}, \dots, \delta_{n,z})$, setting $(h_{i,\sigma})_z := \mathrm{wt}_\sigma(\delta_{i,z})$, for all $\sigma \in S$ we have that

$$\forall 0 \le m \le d_\sigma, \forall n+1-i_{m+1} \le j < n+1-i_m, \forall 0 \le m' < m, \forall n+1-i_{m'+1} \le k < n+1-i_{m'}$$

$$(h_{j,\sigma})_z \notin \left\{ (h_{k,\sigma})_z + a \middle| a \in \mathbb{N}, 0 \le a \le \sum_{r=m'+1}^m k_{\sigma,i_r} \right\}$$

in $k(z)$. On the other hand, for $-\mathbf{k} = (-k_{\sigma,i}) \in \mathbb{Z}_{\le 0}^{\sum_{\sigma \in S} d_\sigma}$, let

$$\bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,\left[-\sum_{r=1}^m k_{\sigma,i_r},0\right]} \subset \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}} \subset \bigcap_{\sigma \in S} \bigcap_{m=1}^{d_\sigma} \mathfrak{X}_n^{\sigma\text{-wu},i_m,[-k_{\sigma,i_m},0]}$$

be the substack of $\mathfrak{X}_n^{S\text{-wu}}$ given by the condition that $D \in \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}(A)$ if and only if for any $z \in \mathrm{Sp}(A)$ and for any triangulation on D_z with parameters $(\delta_{1,z}, \dots, \delta_{n,z})$, setting $(h_{i,\sigma})_z := \mathrm{wt}_\sigma(\delta_{i,z})$, for all $\sigma \in S$ we have that

$$\forall 0 \le m \le d_\sigma, \forall n+1-i_{m+1} \le j < n+1-i_m, \forall 0 \le m' < m, \forall n+1-i_{m'+1} \le k < n+1-i_{m'}$$

$$(h_{j,\sigma})_z \notin \left\{ (h_{k,\sigma})_z - a \middle| a \in \mathbb{N}, 0 \le a \le \sum_{r=m'+1}^m k_{\sigma,i_r} \right\}$$

in $k(z)$.

(iii) When $S = \Sigma_K$ and $I_\sigma = \{1, \dots, n\}$ for all σ, we simply write $\mathfrak{X}_n^{\mathrm{wu},\mathbf{k}}$ to ease the notation.

Remark 4.7. One should think of the conditions defining $\mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}$ as the requirements on weights such that changing the weights through the pullback $\prod_{\sigma\in S, i\in I_\sigma} p_{i,\sigma}^{k_{\sigma,i}}$ do not meet any of the relevant walls in the weight space.

Corollary 4.8. *The following two classes of (φ, Γ_K)-modules of rank n over $\mathcal{R}_{K,A}$ give rise to* $\mathrm{Sp}(A)$*-valued points of $\mathfrak{X}_n^{\mathrm{wu}}$:*

(i) if $D \in \mathfrak{X}_n(A)$ is trianguline and D_z is non-critical crystabelline at all $z \in \mathrm{Sp}(A)$.

(ii) if $D \in \mathfrak{X}_n(A)$ has a strongly non-split triangulation $\mathrm{Fil}^\bullet(D)$ with parameters in $\mathcal{T}_\circ^n(A)$.

Proof. (i) follows from Definition 3.13, and (ii) follows from Corollary 3.5. □

4.2. **Operation $p_{i,\sigma}$ on $\mathfrak{X}_n^{\sigma\text{-wu},i}$.** We begin by recalling Wu's result obtained in [Wu] that we rely on, and then explain how it extends $p_{i,\sigma}$ and allows us to define morphisms on our stacks.

Lemma 4.9. *(i) Let R be any commutative unital ring, and let $Q, S, Q', S' \in R[T]$ be monic polynomials such that $\deg(Q) = \deg(Q'), \deg(S) = \deg(S')$, $Q(T)S(T) = Q'(T)S'(T)$ and $(Q', S) = (1)$. Then, $Q = Q'$ and $S = S'$.*

(ii) Let k be a field of characteristic 0, and let $Q, S, Q', S' \in k[T]$ be monic polynomials such that $QS = Q'S'$ and $Q(T-1)S(T) = Q'(T-1)S'(T)$. Then, $Q = Q'$ and $S = S'$.

(iii) Let A be an affinoid $\mathbb{Q}_p$-algebra, and let $Q, S \in A[T]$ be polynomials. Then, $(Q(T), S(T)) = (1)$ if and only if for any $z \in \mathrm{Sp}(A)$, the sets of roots of $Q(T) \otimes_A k(z)$ and of $S(T) \otimes_A k(z)$ in $\overline{k(z)}$ have empty intersection.

Proof. (i) Subtracting $Q'(T)S(T)$ from both sides of $Q(T)S(T) = Q'(T)S'(T)$, we get

$$(Q - Q')S = Q'(S' - S).$$

Since $(Q', S) = 1$, we find polynomials $A, B \in R[T]$ such that $AQ' + BS = 1$. Multiplying both sides of $AQ' + BS = 1$ by $(S' - S)$, we get

$$AQ'(S' - S) + BS(S' - S) = S' - S.$$

Each of the two terms on the left is divisible by S, so is the right-hand side $S' - S$. Thus there exists $C \in R[T]$ such that $S' - S = CS$, or equivalently,

$$S' = (1 + C)S.$$

But by assumption, S and S' are monic polynomials of the same degree $\deg(S) = \deg(S')$. So, $C = 0$ and $S = S'$. Then $(Q - Q')S = 0$ in $R[T]$. Since S is monic, we have $Q = Q'$.

(ii) The relation $QS = Q'S'$ implies $\frac{Q}{Q'} = \frac{S'}{S} \in k(T)$. The relation $Q(T-1)S(T) = Q'(T-1)S'(T)$ implies $\frac{Q(T-1)}{Q'(T-1)} = \frac{S'(T)}{S(T)}$. Thus, the rational function $C(T) := Q(T)/Q'(T) \in k(T)$ satisfies $C(T) = C(T-1)$, and hence $C(T) = C(T-1) = C(T-2) = C(T-3) = \dots$. Since k has characteristic 0, the rational function $C(T) = Q(T)/Q'(T)$ equals a constant $c \in k$. Then, $c = 1$ since both Q and Q' are monic. Thus, $1 = \frac{Q}{Q'} = \frac{S'}{S}$ as desired.

(iii) This is [Wu, Lemma 3.15]. □

Let D be a (φ, Γ_K)-module of rank n over $\mathcal{R}_{K,A}$. Fix an embedding $\sigma \in \Sigma_K$ and denote by $P_{\mathrm{Sen}}(T) \in \left(K \otimes_{\mathbb{Q}_p} A\right)[T]$ the Sen polynomial of D, and by $P_{\mathrm{Sen},\sigma}(T)$ the σ-Sen polynomial, i.e., the σ-component of $P_{\mathrm{Sen}}(T)$ via $(K \otimes_{\mathbb{Q}_p} A)[T] \simeq \prod_{\sigma\in\Sigma_K} A[T]$.

Theorem 4.10 ([Wu, Proposition 3.16])**.** *Suppose that the σ-Sen polynomial $P_{Sen,\sigma}(T)$ of D admits a decomposition $P_{Sen,\sigma}(T) = Q(T)S(T)$ in $A[T]$ by co-maximal monic polynomials.*

(i) There exists a unique (φ, Γ_K)-module D' over $\mathcal{R}_{K,A}$ contained in D and containing $t_\sigma D$ such that the Sen polynomial of D' is equal to

$$Q(T-1)S(T)\prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma_K} A[T].$$

(ii) There exists a unique (φ, Γ_K)-module D'' over $\mathcal{R}_{K,A}$ contained in $t_\sigma^{-1}D$ and containing D such that the Sen polynomial of D'' is equal to

$$Q(T+1)S(T)\prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma_K} A[T].$$

(iii) If the image of each difference between the roots of $Q(T)$ and $S(T)$ in $\overline{k(z)}$ never belongs to $\{-1, 0, 1\}$ for all $z \in \mathrm{Sp}(A)$, then the operations in (i) and (ii) are mutual inverses.

Proof. (i) This part is [Wu, Proposition 3.16] except that it is stated there that the submodule D' is containing tD, instead of $t_\sigma D$. Let us sketch Wu's construction and the proof of its uniqueness, and then indicate why his proof shows that $t_\sigma D \subset D' \subset D$, which is *a priori* stronger than $tD \subset D' \subset D$.

By Beauville-Laszlo gluing [Wu, Proposition A.3], it suffices to look at the σ-component $D^+_{\mathrm{dif},\sigma}(D)$ of the "localization" $D^+_{\mathrm{dif}}(D) := D \otimes_{\mathcal{R}_{K,A}} (K_\infty \otimes_{\mathbb{Q}_p} A)[[t]]$, which is finite projective of rank n over $(K_\infty \otimes_{K,\sigma} A)[[t]]$ with a semilinear Γ_K-action, and show that $D^+_{\mathrm{dif},\sigma}(D)$ has a unique Γ_K-stable projective submodule M containing $tD^+_{\mathrm{dif},\sigma}(D)$ such that the Sen operator Θ_σ on $D_{\mathrm{Sen},\sigma}(M) = M/tM$ has σ-Sen polynomial $Q(T-1)S(T)$. Then, the (φ, Γ_K)-submodule D' corresponds to the Γ_K-stable lattice in $D^+_{\mathrm{dif}}(D)[1/t]$ given by

$$D^+_{\mathrm{dif},\tau}(D') = \begin{cases} D^+_{\mathrm{dif},\tau}(D) & \text{if } \tau \in \Sigma_K \setminus \{\sigma\}, \\ M & \text{if } \tau = \sigma. \end{cases}$$

This submodule M is explicitly given by

$$M := \ker(D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/t \to \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))),$$

where we have the canonical decomposition as $A[T]$-module on which T acts by Θ_σ,

$$D_{\mathrm{Sen},\sigma}(D) = \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)),$$

thanks to the assumption that $P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T)$ and $(Q(T), S(T)) = 1$.

The uniqueness also follows from this canonical decomposition of $D_{\mathrm{Sen},\sigma}(D)$. Indeed, let $R := K_\infty \otimes_{K,\sigma} A$ and let M be any Γ_K-stable projective $R[[t]]$-submodule M of $D^+_{\mathrm{dif},\sigma}(D)$ containing $tD^+_{\mathrm{dif},\sigma}(D)$ so that the Sen operator Θ_σ on $D_{\mathrm{Sen},\sigma}(M) = M/tM$ has characteristic polynomial $Q(T-1)S(T)$. Then from

$$t^2 D^+_{\mathrm{dif},\sigma}(D) \subset tM \subset tD^+_{\mathrm{dif},\sigma}(D) \subset M \subset D^+_{\mathrm{dif},\sigma}(D),$$

we obtain two exact sequences of Γ_K-stable R-modules

$$0 \to tD^+_{\mathrm{dif},\sigma}(D)/tM \to M/tM \to M/tD^+_{\mathrm{dif},\sigma}(D) \to 0, \tag{4.2.1}$$

$$0 \to M/tD^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/M \to 0, \tag{4.2.2}$$

which, as we now show, following [Zhu17, Lemma 1.1.5], consist of projective R-modules. Since M is projective over $R[[t]]$, M/tM is projective over $R = R[[t]]/tR[[t]]$. Similarly, $D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)$ is projective over R. Since M is an $R[[t]]$-lattice in the projective $R((t))$-module $D_{\mathrm{dif},\sigma}(D) := D^+_{\mathrm{dif},\sigma}(D)[1/t]$ of rank n, it follows that

$$D_{\mathrm{dif},\sigma}(D)/M = \bigoplus_{k\geq 0} t^{-(k+1)}M/t^{-k}M$$

is projective over R since M/tM and so $t^{-(k+1)}M/t^{-k}M$ are projective. Similarly,

$$D_{\mathrm{dif},\sigma}(D)/D^+_{\mathrm{dif},\sigma}(D) = \bigoplus_{k\geq 0} t^{-(k+1)}D^+_{\mathrm{dif},\sigma}(D)/t^{-k}D^+_{\mathrm{dif},\sigma}(D)$$

is projective over R because $D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)$ is projective over R. Hence, from

$$0 \to D^+_{\mathrm{dif},\sigma}(D)/M \to D_{\mathrm{dif},\sigma}(D)/D^+_{\mathrm{dif},\sigma}(D) \to D_{\mathrm{dif},\sigma}(D)/M \to 0$$

we conclude that $D^+_{\mathrm{dif},\sigma}(D)/M$ is projective. By (4.2.2), $M/tD^+_{\mathrm{dif},\sigma}(D)$ is projective. For the sequences (4.2.1) and (4.2.2) of finite projective Γ_K-stable R-modules, we look at the derivative of Γ_K-action, i.e., the Sen operator Θ_σ, to get the factorizations

$$P_{\mathrm{Sen},M/tM}(T) = P_{\mathrm{Sen},tD^+_{\mathrm{dif},\sigma}(D)/tM}(T)\cdot P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T)$$
$$P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/tD^+_{\mathrm{dif},\sigma}(D)}(T) = P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T)\cdot P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T)$$

Since the derivative of Γ_K on $D^+_{\mathrm{dif},\sigma}(D)$ is a derivation $\nabla : D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)$ over the derivation $t\frac{\partial}{\partial t}$ on the coefficient $R[[t]]$, cf. [Fon04, §3.4], it follows that

$$\nabla(ty) = ty + t\nabla y$$

for any $y \in D^+_{\mathrm{dif},\sigma}(D)$, from which we see

$$P_{\mathrm{Sen},tD^+_{\mathrm{dif},\sigma}(D)/tM}(T) = P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T-1).$$

Thus $Q'(T) := P_{\mathrm{Sen},D^+_{\mathrm{dif},\sigma}(D)/M}(T)$, $S'(T) := P_{\mathrm{Sen},M/tD^+_{\mathrm{dif},\sigma}(D)}(T)$ are monic polynomials in $A[T]$ satisfying the equalities

$$Q'(T)S'(T) = P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T),$$
$$Q'(T-1)S'(T) = P_{\mathrm{Sen},M/tM}(T) = Q(T-1)S(T).$$

For each $z \in \mathrm{Sp}(A)$, we can specialize to the polynomial ring $k(z)[T]$ over the residue field $k(z)$ and apply Lemma 4.9(ii) to deduce that $Q_z(T) = Q'_z(T)$ and $S_z(T) = S'_z(T)$. Hence, $\deg(Q') = \deg(Q)$ and $\deg(S) = \deg(S')$, and by Lemma 4.9(iii), we deduce comaximality $(Q, S) = (Q', S) = (Q, S') = (Q', S') = A[T]$, and that $Q'(T) = Q(T)$ and $S'(T) = S(T)$ in $A[T]$ by Lemma 4.9(i). By Chinese remainder theorem, we have a natural isomorphism

$$A[T] \xrightarrow[\sim]{f} A[T]/Q(T) \times A[T]/S(T)$$

and we let $e_Q = f^{-1}(1, 0)$ and $e_S = f^{-1}(0, 1)$ be the idempotents in $A[T]$. Then, for any R-linear section $s' : D^+_{\mathrm{dif},\sigma}(D)/M \to D^+_{\mathrm{dif},\sigma}(D)/t$ that splits (4.2.2) as R-modules, we get an $R[\Theta_\sigma]$-linear section $s := e_Q(\Theta_\sigma) \circ s'$ that splits (4.2.2) as Sen-modules by the choice of e_Q and the fact that $Q' = Q$ and $S = S'$. This must be the canonical decomposition

$$D_{\mathrm{Sen},\sigma}(D) = \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)).$$

So, $M/tD^+_{\mathrm{dif},\sigma}(D) = \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$ and

$$M = \ker(D^+_{\mathrm{dif},\sigma}(D) \to D^+_{\mathrm{dif},\sigma}(D)/t \to \ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))),$$

which proves the uniqueness of such Γ_K-stable projective submodule.

By construction, $tD^+_{\mathrm{dif}}(D) \subset D^+_{\mathrm{dif}}(D') \subset D^+_{\mathrm{dif}}(D)$. Hence, we know that $tD \subset D' \subset D$ by Beauville-Laszlo. To show that $t_\sigma D \subset D' \subset D$, it suffices to note[6] that

- t_σ acts as a unit on the components $D^+_{\mathrm{dif},\tau}(D)$ for $\tau \neq \sigma$, and
- t_σ acts as t on the σ-component $D^+_{\mathrm{dif},\sigma}(D)$.

This means that in the construction of $M \subset D^+_{\mathrm{dif},\sigma}(D)$ above and the proof of uniqueness of such M, we may replace t by t_σ and obtain the same result.

(ii) This is a corollary to (i), after replacing D by $t_\sigma^{-1}D$ and switching the roles played by $Q(T)$ and $S(T)$ in the statement of (i): the Sen polynomial of $t_\sigma^{-1}D$ is exactly

$$P_{\mathrm{Sen},\sigma}(T+1)\prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) = Q(T+1)S(T+1)\prod_{\tau\neq\sigma} P_{\mathrm{Sen},\tau}(T) \in \prod_{\sigma\in\Sigma_K} A[T]$$

The σ-component of $D^+_{\mathrm{dif}}(t_\sigma^{-1}D) = t_\sigma^{-1}D^+_{\mathrm{dif}}(D)$ contains the Γ_K-subrepresentation

$$M = \ker(D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) \to D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D)/t \to \ker(S(\Theta_\sigma+1)|D_{\mathrm{Sen},\sigma}(t_\sigma^{-1}D)))$$

which is the unique Γ_K-stable projective submodule containing

$$tD^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) = t_\sigma D^+_{\mathrm{dif},\sigma}(t_\sigma^{-1}D) = D^+_{\mathrm{dif}}(D)$$

such that Θ_σ acts on $D_{\mathrm{Sen},\sigma}(M)$ with Sen polynomial

$$Q(T+1)S(T+1-1) = Q(T+1)S(T)$$

by (i). The result then follows.

(iii) The assumption ensures $(Q(T), S(T)) = 1 = (Q(T-1), S(T)) = (Q(T+1), S(T))$. The desired conclusion follows from the uniqueness part of (i) and (ii). □

Proposition 4.11. *Let $(D, \mathrm{Fil}^\bullet(D)) \in \mathfrak{X}_B(A)$ be a triangulated (φ, Γ_K)-module over $\mathcal{R}_{K,A}$ with parameters $(\delta_1, \ldots, \delta_n)$. Then,*

$$P_{\mathrm{Sen},\sigma}(T) = \prod_{i=1}^n (T - \mathrm{wt}_\sigma(\delta_i)) \in A[T]$$

factors naturally. Suppose there exists a nonempty subset $I \subset \{1, \ldots, n\}$ such that

$$Q_I(T) := \prod_{m\in I}(T - \mathrm{wt}_\sigma(\delta_m)), \quad S_I(T) := \prod_{m\in\{1,\ldots,n\}\setminus I}(T - \mathrm{wt}_\sigma(\delta_m))$$

[6] One way to see them is to apply the uniqueness proven in the preceding paragraph to $\mathcal{R}_{K,E}$, which yields $t_\sigma\mathcal{R}_{K,E}$ as it has the correct Sen polynomial and satisfies $t\mathcal{R}_{K,E} \subset t_\sigma\mathcal{R}_{K,E} \subset \mathcal{R}_{K,E}$. Comparing $D^+_{\mathrm{dif},\bullet}$ yields

$$t_\sigma(K_\infty \otimes_{K,\tau} E[[t]]) = \begin{cases} K_\infty \otimes_{K,\tau} E[[t]] & \text{if } \tau \neq \sigma, \\ t(K_\infty \otimes_{K,\tau} E[[t]]) & \text{if } \tau = \sigma, , \end{cases}$$

which implies the desired result.

are comaximal in $A[T]$. Then, applying Theorem 4.10(i) (resp. Theorem 4.10(ii)) to

$$P_{\mathrm{Sen},\sigma}(T)=Q_I(T)S_I(T)$$

produces a (φ,Γ_K)-module $D'\subset D$ (resp. $D''\supset D$) that is again trianguline.

Proof. Passing to a finite admissible cover $\{\mathrm{Sp}(A_l)\}_{l=1}^r$ of $\mathrm{Sp}(A)$, we may assume that the line bundles $\mathcal{L}_i=H^0_{\varphi,\Gamma_K}(\mathrm{gr}^i(\mathrm{Fil}^\bullet(D))(\delta_i^{-1}))\cong \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_i),\mathrm{gr}^i(\mathrm{Fil}^\bullet(D)))$ are trivial. For a (φ,Γ_K)-module $R_{K,A}(\delta)$ of rank 1, $D^+_{\mathrm{dif}}(\delta)$ and $D_{\mathrm{Sen}}(\delta)=D^+_{\mathrm{dif}}(\delta)/t$ are free of rank 1. By the functoriality of D^+_{dif}, $D^+_{\mathrm{dif}}(D)$ is free of rank n over $(K_\infty\otimes_{\mathbb{Q}_p}A)[[t]]$. The triangulation $\mathrm{Fil}^\bullet(D)$ induces a basis $\{e_1,\dots,e_n\}$ of $D^+_{\mathrm{dif}}(D)$ such that $\mathrm{Span}(e_1,\dots,e_m)=D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$ for $1<m<n$. Using $D^+_{\mathrm{dif}}(D)=\prod_{\tau\in\Sigma_K}D^+_{\mathrm{dif},\tau}(D)$ and $D_{\mathrm{Sen}}(D)=D^+_{\mathrm{dif}}(D)/t=\prod_{\tau\in\Sigma_K}(D^+_{\mathrm{dif},\tau}(D)/t)=\prod_{\tau\in\Sigma_K}(D^+_{\mathrm{dif},\tau}(D)/t_\tau)=\prod_{\tau\in\Sigma_K}D_{\mathrm{Sen},\tau}(D)$, we get an induced basis $\{e_{1,\tau},\dots,e_{n,\tau}\}$ of $D^+_{\mathrm{dif},\tau}(D)$, which modulo t_τ reduces to a basis $\{\overline{e}_{1,\tau},\dots,\overline{e}_{n,\tau}\}$ of $D^+_{\mathrm{Sen},\tau}(D)$, for each $\tau\in\Sigma_K$.

By the comaximal factorization $P_{\mathrm{Sen},\sigma}(T)=Q_I(T)S_I(T)$, we have the decomposition

$$D_{\mathrm{Sen},\sigma}(D)=\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))\oplus\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)).$$

Thanks to the triangulation, we have the decomposition of $\Theta_\sigma|_{\mathrm{Fil}^m(D)}$ on the filtration steps

$$D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^mD)=\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^mD))\oplus\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^mD))$$

compatible with the filtration, which implies that

$$\mathrm{Fil}^mD_{\mathrm{Sen},\sigma}(D)=\mathrm{Fil}^m\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))\oplus\mathrm{Fil}^m\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$$

for each $1\le m\le n$. We can choose a new basis $\{e'_1,\dots,e'_n\}$ of $D^+_{\mathrm{dif}}(D)$ such that

(1) $\mathrm{Span}(e'_1,\dots,e'_m)=D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$ for $1\le m\le n$, and
(2) we have

$$\mathrm{Fil}^m(\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))=\mathrm{Span}\left\{\overline{e}'_{i,\sigma}\middle|i\in\{1,\dots,m\}\cap I\right\},$$

$$\mathrm{Fil}^m(\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))=\mathrm{Span}\left\{\overline{e}'_{i,\sigma}\middle|i\in\{1,\dots,m\}\setminus I\right\},$$

for each $1\le m\le n$.

Indeed, we obtain $\{e'_1,\dots,e'_n\}$ from $\{e_1,\dots,e_n\}$ by induction on n. When $n=1$, e_1 reduces to an eigenvector $\overline{e}_{1,\sigma}$ of Θ_σ with eigenvalue being the unique root of $P_{\mathrm{Sen},\mathrm{Fil}^1(D)_\sigma}(T)$. So, we can set $e'_1:=e_1$, which satisfies (1) and (2) for the (φ,Γ_K)-module $\mathrm{Fil}^1(D)$ of rank 1. Suppose that, for $1<m\le n$, we have chosen $\{e'_1,\dots,e'_{m-1}\}$ that satisfies (1) and (2) for the (φ,Γ_K)-module $\mathrm{Fil}^l(D)$ of rank l, for all $l\le m-1$. For $\mathrm{Fil}^m(D)$, the basis $\{e'_1,\dots,e'_{m-1},e_m\}$ satisfies (1) but not necessarily (2). Choosing $B,C\in A[T]$ such that $BQ_I+CS_I=1$, we have

$$\overline{e}_{m,\sigma}=(BQ_I+CS_I)(\Theta_\sigma)(\overline{e}_{m,\sigma})=\underbrace{BQ_I(\Theta_\sigma)(\overline{e}_{m,\sigma})}_{\in\ker(S_I(\Theta_\sigma))}+\underbrace{CS_I(\Theta_\sigma)(\overline{e}_{m,\sigma})}_{\in\ker(Q_I(\Theta_\sigma))},$$

and there are two cases for us to discuss:

- If $m\in I$, then $\mathrm{Fil}^m(\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))=\mathrm{Fil}^{m-1}(\ker(S_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))$, which has a basis given by $\left\{\overline{e}'_{i,\sigma}\middle|i\in\{1,\dots,m-1\}\setminus I\right\}$ by the induction hypothesis. Thus,

$$BQ_I(\Theta_\sigma)(\overline{e}_{m,\sigma})=\sum_{i\in\{1,\dots,m-1\}\setminus I}a_{i,\sigma}\overline{e}'_{i,\sigma}$$

belongs to $\mathrm{Span}_{K_\infty\otimes_{K,\sigma}A}\left\{\overline{e}'_{i,\sigma}\middle|i\in\{1,\dots,m-1\}\setminus I\right\}$, for some $a_{i,\sigma}\in K_\infty\otimes_{K,\sigma}A$. Let $a_i\in K_\infty\otimes_{\mathbb{Q}_p}A\cong\prod_{\tau\in\Sigma_K}K_\infty\otimes_{K,\tau}A$ be the element $(0,\dots,0,a_{i,\sigma},0,\dots0)$, and set

$$e'_m:=e_m-\sum_{i\in\{1,\dots,m-1\}\setminus I}a_ie'_i\,.$$

Then, $\{e'_1,\dots,e'_m\}$ is a basis of $D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$, and $\overline{e}'_{m,\sigma}=\overline{e}_{m,\sigma}-BQ_I(\Theta_\sigma)(\overline{e}_{m,\sigma})=CS_I(\Theta_\sigma)(\overline{e}_{m,\sigma})\in\mathrm{Fil}^m(\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))$. Hence, (1) and (2) are satisfied.

- If $m\notin I$, then $\mathrm{Fil}^m(\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))=\mathrm{Fil}^{m-1}(\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))$, which has a basis given by $\left\{\overline{e}'_{i,\sigma}\middle|i\in\{1,\dots,m-1\}\cap I\right\}$ by the induction hypothesis. Similar to the above case when $m\in I$, we can subtract from e_m a suitable $(K_\infty\otimes_{\mathbb{Q}_p}A)$-linear combination of $e'_1,\dots,e'_{m-1}$ to get e'_m such that $\{e'_1,\dots,e'_m\}$ satisfies (1) and (2).

By induction, we conclude the existence of a basis of $D^+_{\mathrm{dif}}(D)$ satisfying (1) and (2). Without loss of generality, we may assume that our chosen basis $\{e_1,\dots,e_n\}$ satisfies (1) and (2).

By the construction of $D''\supset D$, it suffices to consider the submodule $D'\subset D$. By Wu's construction given in the proof of Theorem 4.10, the submodule D' has $D^+_{\mathrm{dif},\tau}(D')=D^+_{\mathrm{dif},\tau}(D)$ for all $\tau\in\Sigma_K\setminus\{\sigma\}$, and using our modified basis, it is easy to see that

$$D^+_{\mathrm{dif},\sigma}(D')=\ker(D^+_{\mathrm{dif},\sigma}(D)\to D^+_{\mathrm{dif},\sigma}(D)/t\to\ker(Q_I(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)))=\mathrm{Span}\{e'_1,\dots,e'_n\},$$

where $e'_m:=t_\sigma e_m$ if $m\in I$ and $e'_m:=e_m$ if $m\notin I$. Since $\mathrm{Span}(e_1,\dots,e_m)=D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D))$ are Γ_K-stable for $1\le m\le n$, the only way for $D^+_{\mathrm{dif},\sigma}(D')$ to be Γ_K-stable is that $\mathrm{Span}(e'_1,\dots,e'_m)$ are Γ_K-stable for $1\le m\le n$. Using the triangulation $\mathrm{Fil}^\bullet(D)$ on D, Beauville-Laszlo gluing [Wu, Proposition A.3] implies that the Γ_K-stable flag on $D^+_{\mathrm{dif}}(D')$

$$\{\mathrm{Span}(e'_1,\dots,e'_m)|1\le m\le n\}$$

determines a triangulation $\mathrm{Fil}^\bullet(D')$ on D' such that $D^+_{\mathrm{dif}}(\mathrm{Fil}^m(D'))=\mathrm{Span}(e'_1,\dots,e'_m)$ and

$$\mathrm{Fil}^m(D')/\mathrm{Fil}^{m-1}(D')\cong\begin{cases}\mathcal{R}_{K,A}(x_\sigma\delta_m) & \text{if } m\in I,\\ \mathcal{R}_{K,A}(\delta_m) & \text{if } m\notin I,\end{cases}$$

for all $1\le m\le n$. $\square$

Theorem 4.12. *(i) Let $D\in\mathfrak{X}_n^{\sigma\text{-wu},i}(A)$. Tate-fpqc locally, choose any triangulation $\mathrm{Fil}^\bullet(D)$ on D over $\mathcal{R}_{K,A}$ with parameters $\delta=(\delta_1,\dots,\delta_n)$ so that*

$$\mathrm{Fil}^i(D)/\mathrm{Fil}^{i-1}(D)\cong\mathcal{R}_{K,A}(\delta_i)\otimes_{\mathcal{O}_{\mathrm{Sp}(A)}}\mathcal{L}_i$$

for some line bundles $\mathcal{L}_i$ on $\mathrm{Sp}(A)$. Over the Tate-fpqc cover, $p_{i,\sigma}(D,\mathrm{Fil}^\bullet(D))$ equals the unique submodule of D defined in Theorem 4.10(i) and is thus independent of the choice of local triangulation $\mathrm{Fil}^\bullet(D)$. The submodule $p_{i,\sigma}(D,\mathrm{Fil}^\bullet(D))$ descends to the unique submodule D' of D over $\mathrm{Sp}(A)$ defined in Theorem 4.10(i).

(ii) Denote the operation $D\mapsto D''$ in Theorem 4.10(ii) by $q_{i,\sigma}$. Then for $D\in\mathfrak{X}_n^{\sigma\text{-wu},i,1}(A)=\mathfrak{X}_n^{\sigma\text{-wu},i,[0,1]}(A)$, we have $p_{i,\sigma}(D)\in\mathfrak{X}_n^{\sigma\text{-wu},i,-1}(A)=\mathfrak{X}_n^{\sigma\text{-wu},i,[-1,0]}(A)$.

(iii) For any subset $S\subset\Sigma_K$, and subsets $I_\sigma\subset\{1,\dots,n\}$ for $\sigma\in S$, and $\mathbf{k}=(k_{i,\sigma})_{\sigma\in S,i\in I_\sigma}\in\mathbb{N}^{\sum_{\sigma\in S}|I_\sigma|}$ any tuple of nonnegative integers, there is an isomorphism

$$p_{\mathbf{k}}:\mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}}\longrightarrow\mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}}$$

given by the composite of elements from $\{p_{i,\sigma}|1 \le i \le n, \sigma \in \Sigma_K\}$ in any order in which $p_{i,\sigma}$ appears with multiplicity $k_{i,\sigma}$, whose inverse is given by

$$q_{\mathbf{k}} : \mathfrak{X}_n^{S\text{-wu},I,-\mathbf{k}} \longrightarrow \mathfrak{X}_n^{S\text{-wu},I,\mathbf{k}}$$

that is defined as the composite of elements from $\{q_{i,\sigma}|1 \le i \le n, \sigma \in \Sigma_K\}$ in any order in which $q_{i,\sigma}$ appears with multiplicity $k_{i,\sigma}$.

Proof. For any (φ, Γ_K)-module D over $\mathrm{Sp}(A)$ that belongs to the stackification of the image of $\beta_B : \mathfrak{X}_B \to \mathfrak{X}_n$, it becomes trianguline after pullback to a Tate-fpqc cover $\{X_j \to \mathrm{Sp}(A)\}_{j \in J}$. The coverings in the Tate-fpqc topology are generated by the usual (admissible) Tate coverings and the morphisms $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ of rigid spaces for faithfully flat maps $A \to B$ of affinoid E-algebras, cf. [EGH, §5.1.7]. Since $\mathrm{Sp}(A)$ is quasi-compact, $\{X_j \to \mathrm{Sp}(A)\}_{j \in J}$ can be refined to an fpqc cover $\mathrm{Sp}(B) \twoheadrightarrow \mathrm{Sp}(A)$. In the proof of (i)-(iii) below, we implicitly take an fpqc morphism $f : \mathrm{Sp}(B) \twoheadrightarrow \mathrm{Sp}(A)$ such that $f^*D = D\widehat{\otimes}_A B$ is trianguline and give our constructions of $p_{i,\sigma}$ and $q_{i,\sigma}$ over $\mathrm{Sp}(B)$. For any point $y \in \mathrm{Sp}(B)$ and $z := f(y) \in \mathrm{Sp}(A)$, the fibers satisfy $(f^*D)_{k(y)} = D_z \otimes_{k(z)} k(y)$. Since $\mathfrak{X}_n$ satisfies Tate-fpqc descent, our constructions on f^*D over $\mathrm{Sp}(B)$ descend to constructions on D over $\mathrm{Sp}(A)$, because the uniqueness part of Theorem 4.10 implies that $p_{i,\sigma}$ and $q_{i,\sigma}$ respect the isomorphisms in the descent data. Thus, $p_{i,\sigma}(f^*D) \subset f^*D$ descends to $p_{i,\sigma}(D) \subset D$ in (i), and $q_{i,\sigma}(f^*D) \supset f^*D$ descends to $q_{i,\sigma}(D) \supset D$ in (ii).

(i) Passing to a finite admissible cover $\{\mathrm{Sp}(A_l)\}_{l=1}^r$ of $\mathrm{Sp}(A)$, we assume the $\mathcal{L}_i$ are trivial. By the definition of U_i, we have that for all $z \in \mathrm{Sp}(A)$,

$$\{\mathrm{wt}_\sigma(\delta_{1,z}), \ldots, \mathrm{wt}_\sigma(\delta_{n-i,z})\} \cap \{\mathrm{wt}_\sigma(\delta_{n-i+1,z}), \ldots, \mathrm{wt}_\sigma(\delta_{n,z})\} = \emptyset.$$

inside the residue field $k(z)$ for all $z \in \mathrm{Sp}(A)$. Thus, for $Q_{i,\sigma}(T) := \prod_{j=n-i+1}^n (T - \mathrm{wt}_\sigma(\delta_j))$ and $S_{i,\sigma}(T) := \prod_{j=1}^{n-i} (T - \mathrm{wt}_\sigma(\delta_j))$, we have $(Q_{i,\sigma}(T), S_{i,\sigma}(T)) = (1)$ and a factorization

$$P_{\mathrm{Sen},\sigma}(T) = Q_{i,\sigma}(T) S_{i,\sigma}(T) \in A[T],$$

which is independent of the choice of $\mathrm{Fil}^\bullet(D)$: indeed, if $\{\delta'_m\}_{m=1}^n$ is the parameters attached to another triangulation of D over $\mathcal{R}_{K,A}$, then we get another co-maximal factorization

$$P_{\mathrm{Sen},\sigma}(T) = Q'_{i,\sigma}(T) S'_{i,\sigma}(T) \in A[T]$$

where we similarly put $Q'_{i,\sigma}(T) := \prod_{j=n-i+1}^n (T - \mathrm{wt}_\sigma(\delta'_j))$ and $S'_{i,\sigma}(T) := \prod_{j=1}^{n-i} (T - \mathrm{wt}_\sigma(\delta'_j)))$. Moreover, we have $(Q'_{i,\sigma}(T), S_{i,\sigma}(T)) = (Q_{i,\sigma}(T), S'_{i,\sigma}(T)) = 1$ by Lemma 4.9(iii). We conclude that $Q_{i,\sigma}(T) = Q'_{i,\sigma}(T)$ and $S_{i,\sigma}(T) = S'_{i,\sigma}(T)$ by Lemma 4.9(i).

Let $\{e_1, \ldots, e_{n-i}, \ldots, e_n\}$ be a basis of $D^+_{\mathrm{dif}}(D)$ as in the proof of Proposition 4.11 with

$$I := \{k | n - i + 1 \le k \le n\} \subset \{1, \ldots, n\}$$

such that for the splitting of $D_{\mathrm{Sen},\sigma}(D)$ as

$$\ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \oplus \ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D)) \cong D_{\mathrm{Sen},\sigma}(D/\mathrm{Fil}^{n-i}(D)) \oplus D_{\mathrm{Sen},\sigma}(\mathrm{Fil}^{n-i}(D))$$

according to the derivative Θ_σ of the Γ_K-action, the mod-t_σ reduction $\{\overline{e_1}, \ldots, \overline{e_{n-i}}\}$ is a basis of $\ker(S(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$ and $\{\overline{e_{n-i+1}}, \ldots, \overline{e_n}\}$ is a basis of $\ker(Q(\Theta_\sigma)|D_{\mathrm{Sen},\sigma}(D))$. From the proof of Proposition 4.11, we see that the submodule N of basis

$$\{e_1, \ldots, e_{n-i}, t_\sigma e_{n-i+1}, \ldots, t_\sigma e_n\}$$

in $D^+_{\mathrm{dif}}(D)$ corresponds to the module M given by Wu in Theorem 4.10(i). Thus $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ corresponds to M since the localization of $p_{i,\sigma}(D, \mathrm{Fil}^\bullet(D))$ clearly has the same basis

$$\{e_1, \dots, e_{n-i}, t_\sigma e_{n-i+1}, \dots, t_\sigma e_n\}$$

by the definition of pullback, and (i) is proven.

(ii) Keep the notations in the proof of (i), in particular the basis $e_1, \dots, e_n$. By (i), $q_{i,\sigma}(D)$ can be computed using any triangulation $\mathrm{Fil}^\bullet(D)$ on D with respect to which our basis is chosen, because all triangulations induce the same factorization of σ-Sen polynomial. We know that $q_{i,\sigma}(D)$ is trianguline by Proposition 4.11. Let $h_1, \dots, h_n$ be the ordered σ-Sen weights of D with respect to any triangulation $\mathrm{Fil}^\bullet(D)$. Then,

$$h'_m = \begin{cases} h_m & \text{if } 1 \le m \le n-i, \\ h_m + 1 & \text{if } n-i+1 \le m \le n. \end{cases}$$

are the ordered σ-Sen weights of an induced triangulation on the pullback $D' := p_{i,\sigma}(D)$. Since we assume $D \in \mathfrak{X}_n^{\sigma\text{-wu},i,1}(A) = \mathfrak{X}_n^{\sigma\text{-wu},i,[0,1]}(A)$, we have

$$h'_j \not\equiv h'_k \pmod{\mathfrak{m}_z}, \quad h'_j \not\equiv h'_k - 1 \pmod{\mathfrak{m}_z}$$

for all $z \in \mathrm{Sp}(A)$ with the corresponding maximal ideal $\mathfrak{m}_z$ and for all $1 \le j \le n-i < k \le n$. In particular, we can apply $q_{i,\sigma}$ to D'. It remains to show that D' is σ-weight-uniform. Assume, for the sake of contradiction, that there exists another triangulation on D' such that at some point $z \in \mathrm{Sp}(A)$, the induced ordering on σ-Sen weights is

$$(h'_{w(1),z}, \dots, h'_{w(n),z})$$

for some $w \in S_n$ such that for some $1 \le m_0 \le n$, $h'_{w(m_0),z} \neq h'_{m_0,z} \in k(z)$. By Proposition 4.11, this other triangulation on D' induces on $q_{i,\sigma}(D') = q_{i,\sigma}(p_{i,\sigma}(D)) = D$ a triangulation whose ordered σ-Sen weights are $(h''_1, \dots, h''_n)$ with

$$h''_m = \begin{cases} h'_{w(m)} - 1 = h_{w(m)} & \text{if } w(m) \in I := \{k | n-i+1 \le k \le n\}, \\ h'_{w(m)} = h_{w(m)} & \text{if } w(m) \notin I. \end{cases}$$

<u>Case (a)</u>. If $\{m_0, w(m_0)\}$ is contained in either I or I^c, then $h'_{w(m_0),z} \neq h'_{m_0,z} \in k(z)$ implies

$$h''_{m_0,z} = h_{w(m_0),z} \neq h_{m_0,z} \in k(z)$$

contrary to the assumption that D is σ-weight-uniform.
<u>Case (b)</u>. Otherwise, either $m_0 \in I$ but $w(m_0) \notin I$, or $m_0 \notin I$ but $w(m_0) \in I$. Then we have

$$h''_{m_0,z} = h_{w(m_0),z} \neq h_{m_0,z} \in k(z)$$

because $D \in \mathfrak{X}_n^{\sigma\text{-wu},i}(A)$, contrary to the assumption that D is σ-weight-uniform.

Hence, we conclude that $p_{i,\sigma} : \mathfrak{X}_n^{\sigma\text{-wu},i,1} \to \mathfrak{X}_n^{\sigma\text{-wu},i,-1}$ preserves σ-weight-uniformity.

(iii) By Lemma 4.2, the independence (i) of triangulations for the pullback interpretation of $p_{i,\sigma}$ under the given assumption on weights, and (ii) that $p_{i,\sigma}$ preserves weight-uniformity under the given assumption on weights, it follows that any ways of composing $p_{i,\sigma}$ with multiplicity $k_{i,\sigma}$ as i varies in $\{1, \dots n\}$ and σ varies in Σ_K produce the same result, which is the morphism $p_{\mathbf{k}}$. Since $q_{i,\sigma}$ is the inverse of $p_{i,\sigma}$ by Theorem 4.10(iii), it follows that we can define $q_{\mathbf{k}}$ by composing $q_{i,\sigma}$'s with multiplicity $k_{i,\sigma}$ in any order. It is clear that $p_{\mathbf{k}}$ and $q_{\mathbf{k}}$ are mutual inverses. □

4.2.1. To summarize, we have $p_{i,\sigma}: \mathfrak{X}_B \to \mathfrak{X}_B$, which is often invertible by Theorem 3.1. By Theorem 4.12, this induces a "change of weights" morphism of stacks

$$p_{i,\sigma}: \mathfrak{X}_n^{\sigma\text{-wu},i} \to \mathfrak{X}_n$$

which, if the "regularity" of the Sen weights is unchanged after the weight change, is invertible:

$$p_{i,\sigma}: \mathfrak{X}_n^{\sigma\text{-wu},i,1} \xrightarrow{\sim} \mathfrak{X}_n^{\sigma\text{-wu},i,-1}$$

by Theorem 4.12(iii). Moreover, we know by Lemma 4.2 that for any $1 \le i, j \le n$ and $\tau, \sigma \in \Sigma_K$,

$$(p_{j,\tau} \circ p_{i,\sigma})(D) = (p_{i,\sigma} \circ p_{j,\tau})(D)$$

whenever the pullbacks are independent of the choice of triangulation.

$$p_{\mathbf{k}}: \mathfrak{X}_n^{\mathrm{wu},\mathbf{k}} \xrightarrow{\sim} \mathfrak{X}_n^{\mathrm{wu},-\mathbf{k}}$$

as long as the "regularity" of the Sen weights is unchanged after the change. See [Wu, Example 3.18] for the general (non-invertible) case of changing the Sen weights on $\mathfrak{X}_n$ in a similar way.

One family of "nice" D are those non-critical crystabelline (φ, Γ_K)-modules $D \in \mathfrak{X}_n^{\mathrm{wu}}(A)$ of regular Sen weights. When $A = E$ is a finite extension of $\mathbb{Q}_p$, we verify in the last section that $p_{i,\sigma}$ on D corresponds to the translation functor on $\pi_{\mathrm{fs}}(D)$, for Ding's construction $\pi_{\mathrm{fs}}(D)$.

4.3. **Étaleness.** According to Jean-Marc Fontaine and others, there is an equivalence of categories D_{rig} between continuous E-linear $\mathrm{Gal}(\overline{K}|K)$-representations and étale (φ, Γ_K)-modules over $\mathcal{R}_{K,E}$. We study in this subsection how the pullback operations interplay with étaleness on certain very generic or non-critical crystabelline (φ, Γ_K)-modules.

In this subsection, we write $\mathcal{R} := \mathcal{R}_{K,E}$. Let $\mathcal{C}_E$ be the category of local Artinian E-algebra $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A \cong E$. By [BC09, Lemma 2.2.5], our discussion also applies to A-linear Galois representations and (φ, Γ_K)-modules over $\mathcal{R}_{K,A}$ for $A \in \mathcal{C}_E$.

Let us recall Kedlaya's theory of slopes. For a (φ, Γ_K)-module D of rank n over $\mathcal{R}$, we define its **degree** by $\deg(D) := \deg(\wedge^n D)$, which is the p-adic valuation of a "φ-eigenvalue" on $\wedge^n D$. Then, the **slope** of D is defined by $\mu(D) := \deg(D)/\mathrm{rank}(D)$. We say that D is **semistable** if for any finite free φ-submodule M of D satisfying $\varphi^* M \cong M$, one has $\mu(M) \ge \mu(D)$. Then, D is called **étale** if it is semistable of slope zero, cf. [Liu08, p.8].

Remark 4.13. The following properties clearly hold for (φ, Γ)-modules.

(i) If $0 \to D' \to D \to D'' \to 0$ is exact, then $\deg(D) = \deg(D') + \deg(D'')$.
(ii) One has $\mu(D_1 \otimes D_2) = \mu(D_1) + \mu(D_2)$.
(iii) We have $\deg(D^\vee) = -\deg(D)$ and $\mu(D^\vee) = -\mu(D)$.

Example 4.14. By [KPX14, Construction 6.2.4], for any continuous character $\delta: K^\times \to E^\times$, if we factorize $\delta = \delta^{\mathrm{unr}}\delta^{\mathrm{wt}}$ such that $\delta^{\mathrm{unr}}(\pi_K) = \delta(\pi_K)$ and $\delta^{\mathrm{wt}}|_{\mathcal{O}_K^\times} = \delta|_{\mathcal{O}_K^\times}$, then $\mathcal{R}(\delta) = \mathcal{R}(\delta^{\mathrm{unr}}) \otimes_{\mathcal{R}} \mathcal{R}(\delta^{\mathrm{wt}})$. Since $\mathcal{R}(\delta^{\mathrm{wt}})$ is étale and $\varphi^f = \delta(\pi_K)$ on $\mathcal{R}(\delta^{\mathrm{unr}}) = D_{f,\delta(\pi_K)} \otimes_{K_0 \otimes E} \mathcal{R}$, where $D_{f,\delta(\pi_K)} = K_0 \otimes_{\mathbb{Q}_p} E$ carries a Frobenius semilinear endomorphism φ such that $\varphi^f = \delta(\pi_K)$, cf. [KPX14, Lemma 6.2.3], it follows from Remark 4.13(ii) that

$$\mu(\mathcal{R}(\delta)) = \deg(\mathcal{R}(\delta)) = \frac{1}{f} v_p(\delta(\pi_K)),$$

where v_p is the p-adic valuation on $\overline{\mathbb{Q}_p}$ normalized by the condition $v_p(p) = 1$.

The following is the slope filtration theorem by Kedlaya.

Theorem 4.15 ([Liu08, Theorem 2.4]). *Every (φ,Γ_K)-module D over $\mathcal{R}_{K,E}$ admits a unique filtration $0=D_0\subset D_1\subset\cdots\subset D_l=D$ by saturated (φ,Γ_K)-submodules whose successive quotients are semistable with increasing slopes $\mu(D_1/D_0)<\cdots<\mu(D_l/D_{l-1})$.*

In view of Theorem 4.15 and Remark 4.13(i), we deduce

Corollary 4.16. *A (φ,Γ_K)-module D over $\mathcal{R}_{K,E}$ is étale if and only if $\mu(D)=0$ and it does not contain any saturated (φ,Γ_K)-submodule of strictly negative slope.*

4.3.1. Suppose $(D,\mathrm{Fil}^\bullet(D))$ is strongly non-split of parameters in $\mathcal{T}_\circ^n(E)$. By Corollaries 3.5 and 4.16, D is étale if and only if $\mathrm{Fil}^i(D)$ has non-negative slopes and D has slope zero. So,

Proposition 4.17. *(i) If $(D,\mathrm{Fil}^\bullet(D))$ is strongly non-split of parameter $(\delta_1,\ldots,\delta_n)\in\mathcal{T}_\circ^n(E)$, then D is étale if and only if $\sum_{i=1}^m v_p(\delta_i(\pi_K))\geq 0$ for $1\leq m<n$ and $\sum_{i=1}^n v_p(\delta_i(\pi_K))=0$.*

(ii) If $(D,\mathrm{Fil}^\bullet(D))$ is strongly non-split of parameter $(\delta_1,\ldots,\delta_n)\in\mathcal{T}_\circ^n(E)$ and étale, then $p_{j,\sigma}(D)$ is étale up to twist if and only if

$$\begin{cases}\displaystyle\sum_{i=1}^m v_p(\delta_i(\pi_K))\geq\frac{mj}{ne} & 1\leq m\leq n-j;\\ \displaystyle\sum_{i=1}^m v_p(\delta_i(\pi_K))+\frac{m-(n-j)}{e}\geq\frac{mj}{ne} & n-j<m<n.\end{cases}$$

Proof. (i) By Remark 4.13(i) and the definition of slope,

$$\mu(\mathrm{Fil}^i(D))=\deg(\mathrm{Fil}^i(D))/\mathrm{rank}(\mathrm{Fil}^i(D))=\frac{1}{i}\sum_{j=1}^i\deg(\mathcal{R}(\delta_j))$$

By Example 4.14, we see that $\mu(\mathrm{Fil}^i(D))=\dfrac{1}{fi}\displaystyle\sum_{j=1}^i v_p(\delta_j(\pi_K))$, for all i.

(ii) The new parameter of $p_{j,\sigma}(D)$ is $(\delta_1,\ldots,\delta_{n-j},x_\sigma\delta_{n-j+1},x_\sigma\delta_{n-j+2},\ldots,x_\sigma\delta_n)$, and

$$v_p(x_\sigma(\pi_K))=v_p(\sigma(\pi_K))=v_p(\pi_K)=1/e.$$

Let $\chi:K^\times\to E^\times$ be any continuous character. The twist $p_{j,\sigma}(D)(\chi)$ is again non-split and very generic of parameter $(\delta_1\chi,\ldots,\delta_{n-j}\chi,\ldots x_\sigma\delta_{n-j+1}\chi,x_\sigma\delta_{n-j+2}\chi,\ldots,x_\sigma\delta_n\chi)$, which is étale if and only if we have

$$\begin{cases}\sum_{i=1}^m v_p(\delta_i(\pi_K))+mv_p(\chi(\pi_K))\geq 0 & \text{if } 1\leq m\leq n-j\\ \sum_{i=1}^m v_p(\delta_i(\pi_K))+(m-(n-j))v_p(\pi_K)+mv_p(\chi(\pi_K))\geq 0 & \text{if } n-j<m\leq n-1\\ \sum_{i=1}^n v_p(\delta_i(\pi_K))+jv_p(\pi_K)+nv_p(\chi(\pi_K))=0 & \text{if } m=n\end{cases}$$

by (i) above. Since D is étale, $\sum_{i=1}^n v_p(\delta_i(\pi_K))+jv_p(\pi_K)+nv_p(\chi(\pi_K))=0$ implies that $v_p(\chi(\pi_K))=-j/(ne)$, which allows us to complete the proof by taking unramified χ. □

4.3.2. In this paragraph, we consider the étaleness for $D\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$. By Corollary 4.16 and Theorem 3.10, all saturated (φ,Γ_K)-submodules of D are $\mathrm{Fil}_w^i(D)=D_{w,i}$ as in Proposition 3.14 for $0\leq i\leq n$ and $w\in S_n$. Let $\{\alpha_i:=\phi_i(\pi_K)\}_{1\leq i\leq n}$ be the φ^f-eigenvalues on $D_{\mathrm{cris}}^{K_m}(D)$. Then, by Proposition 3.14(ii) and Example 4.14, we have

$$v_p(\delta_{w,i}(\pi_K))=v_p\left(\prod_\sigma\sigma(\pi_K)^{h_{i,\sigma}}\phi_{w(i)}(\pi_K)\right)=v_p(\alpha_{w(i)})+\sum_\sigma h_{i,\sigma}\frac{1}{e}. \tag{4.3.1}$$

Proposition 4.18. *Suppose $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. Let $\tau \in S_n$ be a refinement such that*

$$v_p(\alpha_{\tau(1)}) \leq v_p(\alpha_{\tau(2)}) \leq \cdots \leq v_p(\alpha_{\tau(n)}).$$

Then, D is étale if and only if

$$\begin{cases} \sum_{i=1}^{j} v_p(\alpha_{\tau(i)}) \geq -\frac{1}{e} \sum_{\sigma} \sum_{i=1}^{j} h_{i,\sigma} & \text{if } 1 \leq j < n, \\ \sum_{i=1}^{n} v_p(\alpha_i) = -\frac{1}{e} \sum_{\sigma,i} h_{i,\sigma} & \text{if } j = n. \end{cases}$$

Proof. It follows immediately from Corollary 4.16, our choice of τ, and Equation (4.3.1). ☐

Remark 4.19. Alternatively, to prove Proposition 4.18 we can observe that, by our choice of τ, the n inequalities amount to having that the Hodge filtration on $D_{\mathrm{cris}}(D)$ is weakly admissible in the sense of [Fon94, Definition 4.4.3], or concretely [BS07, Equations (4) and (5)]. Indeed, by [BM02, Proposition 3.1.1.5] and [Fon94, Proposition 4.4.9], it suffices to check $t_H(D') \geq t_N(D')$ for all E-*filtered* $(\varphi, G(K_m/K))$-submodules over K of $D_{\mathrm{cris}}(D)$ with the induced filtration. But these D' are in bijection with the $n!$ refinements, so their Newton numbers are clear, and their Hodge numbers are clear by the non-criticalness assumption. A computation similar to that for [BS07, Proposition 3.2, $(i) \Rightarrow (ii)$] gives the n inequalities.

Corollary 4.20. *Suppose $D \in \mathfrak{X}_n(E)$ satisfies either the hypothesis of Proposition 4.17 or that of Proposition 4.18 and is étale of regular Sen weights. Suppose that $p_{i,\sigma}(D)$ again satisfies the hypothesis of Proposition 4.17 or that of Proposition 4.18 and is étale after twist by a character $\chi : K^\times \to E^\times$ (whether this is possible can be checked by Proposition 4.17 or 4.18).*

Then, for any local Artinian E-algebra $A \in \mathcal{C}_E$ of residue field E and for any deformation D_A of D to an element in $\mathfrak{X}_n(A)$, $p_{i,\sigma}(D_A)$ is again étale after twisting by the same character χ.

Proof. First of all, by $p_{i,\sigma}(D_A)$ we mean the result of applying Wu's construction (Theorem 4.10) to D_A, which is applicable and deforms $p_{i,\sigma}(D)$ from $\mathcal{R}_{K,E}$ to $\mathcal{R}_{K,A}$ since at the unique closed point $z \in \mathrm{Sp}(A)$, $D_{A,z} = D$ is assumed to have regular Sen weights.

The étaleness of $p_{i,\sigma}(D_A)(\chi)$ follows from the assumption that $p_{i,\sigma}(D)(\chi)$ is étale and the fact that extensions of pure of slope s φ-modules are pure of slope s by [BC09, Lemma 2.2.5]. ☐

5. Relation with translation functors

5.1. **Trianguline deformations of (φ, Γ_K)-modules.** We discuss certain deformations of (φ, Γ_K)-modules from $\mathcal{R}_{K,E}$ to $\mathcal{R}_{K,A}$ when A is a local Artinian E-algebra of residue field E.

Definition 5.1. Let D be a (φ, Γ_K)-module D of rank n over $\mathcal{R}_{K,E}$, and let $A \in \mathcal{C}_E$ be a local Artinian E-algebra with maximal ideal $\mathfrak{m}_A$ such that $A/\mathfrak{m}_A \cong E$.

- (i) By a **trianguline deformation** of D to $\mathcal{R}_{K,A}$ we mean a trianguline (φ, Γ_K)-module D_A over $\mathcal{R}_{K,A}$ such that $D_A \otimes_A E = D_A/\mathfrak{m}_A D_A \cong D$. Note that if $\mathrm{Fil}^\bullet(D_A)$ is a triangulation on D_A, then we have an induced triangulation $\mathrm{Fil}^\bullet(D_A) \otimes_A E$ on $D_A \otimes_A E \cong D$.
- (ii) For any given triangulation $\mathcal{F}^\bullet$ on D, an **$\mathcal{F}^\bullet$-trianguline deformation** of D to $R_{K,A}$ is a pair $(D_A, \mathrm{Fil}^\bullet(D_A))$ consisting of a deformation D_A of D to $\mathcal{R}_{K,A}$ and a triangulation $\mathrm{Fil}^\bullet(D_A)$ such that $\mathrm{Fil}^\bullet(D_A) \otimes_A E = \mathcal{F}^\bullet$ via the map $D_A \twoheadrightarrow D_A/\mathfrak{m}_A D_A \cong D$.

The next lemma generalizes [BC09, Proposition 2.3.6] from the case $K = \mathbb{Q}_p$ to arbitrary finite extension $K/\mathbb{Q}_p$, with essentially the same argument.

Lemma 5.2. *Let D be a (φ, Γ_K)-module D of rank n over $\mathcal{R}_{K,E}$. Let $A \in \mathcal{C}_E$ be a local Artinian E-algebra with maximal ideal $\mathfrak{m}_A$ such that $A/\mathfrak{m}_A \cong E$.*

(i) Suppose that for a continuous character $\delta : K^\times \to E^\times$, D has a saturated (φ, Γ_K)-submodule D_0 of rank 1 such that $D_0 \cong \mathcal{R}_{K,E}(\delta)$ and $\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,E}(\delta), D/D_0) = 0$. Then for any deformation D_A of D to $\mathcal{R}_{K,A}$, there can exist at most one continuous character $\delta_A : K^\times \to A^\times$ such that D_A contains an $\mathcal{R}_{K,E}$-saturated (φ, Γ_K)-submodule $D_{A,0}$ over $\mathcal{R}_{K,A}$ and $(\delta_A \mod \mathfrak{m}_A) : K^\times \to E^\times$ equals δ. Moreover, if such δ_A exists, $D_{A,0}$ is also the unique $\mathcal{R}_{K,E}$-saturated (φ, Γ)-submodule of D_A isomorphic to $\mathcal{R}_{K,A}(\delta_A)$.

(ii) Suppose D is trianguline with a triangulation $\mathcal{F}^\bullet = (0 = \mathcal{F}^0 \subsetneq \mathcal{F}^1 \subsetneq \cdots \subsetneq \mathcal{F}^n = D)$ with parameter $(\delta_1, \ldots, \delta_n)$ such that for all $1 \leq i < j \leq n$, $\delta_i/\delta_j \notin \{x^{\mathbf{k}} \big| \mathbf{k} \in \mathbb{N}^{\Sigma_K}\}$. Then, for any trianguline deformation D_A of D, D_A can admit at most one triangulation $\mathrm{Fil}^\bullet(D_A)$ such that $(D_A, \mathrm{Fil}^\bullet(D_A))$ is an $\mathcal{F}^\bullet$-trianguline deformation of D.

Proof. (i) Given $\delta : K^\times \to E^\times$, let $\delta_A^{(1)}, \delta_A^{(2)} : K^\times \to A^\times$ be two lifts of δ, and suppose $D_{A,0}^{(i)}$ is an $\mathcal{R}_{K,E}$-saturated (φ, Γ_K)-submodule over $\mathcal{R}_{K,A}$ of D_A such that $D_{A,0}^{(i)} \cong \mathcal{R}_{K,A}(\delta_A^{(i)})$, for $i = 1, 2$. We need to show that $D_{A,0}^{(1)} = D_{A,0}^{(2)}$ as submodules of D_A and thus $\delta_A^{(1)} = \delta_A^{(2)}$. It suffices to show that, for any $i, j \in \{1, 2\}$, we have

$$\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_A^{(j)}), D_A/D_A^{(i)}) = 0 \tag{5.1.1}$$

from which we can conclude that $D_A^{(j)} \subset D_A^{(i)}$, and hence by symmetry, $D_A^{(1)} = D_A^{(2)}$.

Since $D_A^{(i)}$ is a free A-submodule of the free A-module D_A, the quotient $D_A/D_A^{(i)}$ is also free over A by [BC09, Lemma 2.2.3(i)]. So, $D_A^{(i)}$ is a direct summand of D_A as A-module, and hence $D_A^{(i)} \cap \mathfrak{m}_A D_A = \mathfrak{m}_A D_A^{(i)}$, by which we see that $D_A^{(i)}/\mathfrak{m}_A D_A^{(i)} \cong \mathcal{R}_{K,E}(\delta)$ is a saturated (φ, Γ_K)-submodule of $D_A/\mathfrak{m}_A D_A \cong D$ over $\mathcal{R}_{K,E}$. Since $\mathrm{Hom}_{\varphi,\Gamma_K}(R_{K,E}(\delta), D/D_0)$ vanishes, $D_A^{(i)}/\mathfrak{m}_A D_A^{(i)}$ is a saturated submodule of D_0 of rank 1. So, $D_A^{(i)}/\mathfrak{m}_A D_A^{(i)} = D_0$. For $M_i := D_A/D_A^{(i)}$, we prove (5.1.1) by dévissage through the $\mathfrak{m}_A$-adic filtration:

$$0 = \mathfrak{m}_A^N M_i \subset \cdots \subset \mathfrak{m}_A^2 M_i \subset \mathfrak{m}_A M_i \subset M_i$$

for some large $N > 0$ such that $\mathfrak{m}_A^N = 0$, whose graded pieces are

$$\mathfrak{m}_A^l M_i/\mathfrak{m}_A^{l+1} M_i \cong M_i \otimes_A (\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1}) \cong (D/D_0)^{\dim_E(\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1})}$$

by the projectivity of M_i over A. By left exactness of $\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_A^{(j)}), -)$, it is enough to show $\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_A^{(j)}), M_i \otimes_A (\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1})) = 0$ for all $0 \leq j < N$. Any nonzero map in $\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,A}(\delta_A^{(j)}), M_i \otimes_A (\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1}))$ factors through $\mathcal{R}_{K,A}(\delta_A^{(j)}) \otimes_A E = \mathcal{R}_{K,E}(\delta)$, which contradicts the fact that $\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,E}(\delta), D/D_0) = 0$. So, (5.1.1) is proved.

(ii) We claim that $\mathrm{Fil}^i(D_A)/\mathrm{Fil}^{i-1}(D_A)$ is the unique (φ, Γ_K)-submodule of $D_A/\mathrm{Fil}^{i-1}(D_A)$ that lifts the submodule $\mathcal{F}^i/\mathcal{F}^{i-1}$ of rank 1 in $D/\mathcal{F}^{i-1}$, for all $1 \leq i \leq n$, which is enough for us to conclude the uniqueness of the $\mathcal{F}^\bullet$-triangulation on D_A. Under the assumption that for all $1 \leq i < j \leq n$, $\delta_j/\delta_i \notin \{x^{-\mathbf{k}} \big| \mathbf{k} \in \mathbb{N}^{\Sigma_K}\}$, we deduce from Lemma 2.7(ii) that

$$\begin{aligned}\mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,E}(\delta_i), (D/\mathcal{F}^{i-1})/(\mathcal{F}^i/\mathcal{F}^{i-1})) &= \mathrm{Hom}_{\varphi,\Gamma_K}(\mathcal{R}_{K,E}(\delta_i), D/\mathcal{F}^i) \\ &= H^0((D/\mathcal{F}^i)(\delta_i^{-1})) = 0.\end{aligned}$$

So, we can apply (i) to the deformation $D_A/\mathrm{Fil}^{i-1}(D_A)$ and the character δ_j to conclude the uniqueness of $\mathrm{Fil}^i(D_A)/\mathrm{Fil}^{i-1}(D_A)$ in $D_A/\mathrm{Fil}^{i-1}(D_A)$. □

Remark 5.3. For a (φ,Γ_K)-module D of rank n over $\mathcal{R}_{K,E}$, if the local Artinian E-algebra A is $E[\varepsilon]/\varepsilon^2$, we use $\widetilde{D}$ instead of $D_{E[\varepsilon]/\varepsilon^2}$ to denote the deformations of D to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$. We can identify *elements* in $\mathrm{Ext}^1(D,D)$ with deformations of D over $R_{K,E[\varepsilon]/\varepsilon^2}$ as follows. Let

$$0\to D\xrightarrow{\iota}\widetilde{D}\xrightarrow{\pi}D\to 0$$

be a self-extension of D as (φ,Γ_K)-module over $\mathcal{R}_{K,E}$. We define an action of ε on $\widetilde{D}$ by

$$\varepsilon_{\widetilde{D}}:\widetilde{D}\xrightarrow{\pi}D\xrightarrow{\mathrm{id}}D\xrightarrow{\iota}\widetilde{D}.$$

Since ι and π are (φ,Γ_K)-equivariant, $\varepsilon_{\widetilde{D}}$ commutes with φ and Γ_K. Also, $\varepsilon_{\widetilde{D}}^2=\iota\circ\pi\circ\iota\circ\pi=\iota\circ 0\circ\pi=0$, which shows that $\widetilde{D}$ is a (φ,Γ_K)-module of rank n over $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$ with $\widetilde{D}/\varepsilon\widetilde{D}\cong D$. So, $\widetilde{D}$ is a deformation of D to $R_{K,E[\varepsilon]/\varepsilon^2}$. Conversely, suppose $\widetilde{D}$ is a (φ,Γ_K)-module of rank n over $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$ such that $\widetilde{D}/\varepsilon\widetilde{D}\cong D$. Then, $\varepsilon:\widetilde{D}/\varepsilon\widetilde{D}\to\varepsilon\widetilde{D}$ is an isomorphism of (φ,Γ_K)-module over $\mathcal{R}_{K,E}$ because, as $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$-module, $\widetilde{D}\cong(\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2})^n=(\mathcal{R}_{K,E})^n\otimes_E E[\varepsilon]/\varepsilon^2$. We therefore get a self-extension

$$0\to\varepsilon\widetilde{D}\cong D\hookrightarrow\widetilde{D}\twoheadrightarrow\widetilde{D}/\varepsilon\widetilde{D}\cong D\to 0$$

of D, which is an element in $\mathrm{Ext}^1(D,D)$.

Definition 5.4. For any given triangulation $\mathcal{F}^\bullet$ on a (φ,Γ_K)-module D of rank n over $\mathcal{R}_{K,E}$, we define the **subspace of $\mathcal{F}^\bullet$-trianguline deformations** $\mathrm{Ext}^1_{\mathcal{F}^\bullet}(D,D)$ to be the subspace of $\mathrm{Ext}^1(D,D)$ consisting of deformations $\widetilde{D}$ of D to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$ that admits an $\mathcal{F}^\bullet$-triangulation $\mathrm{Fil}^\bullet(\widetilde{D})$ under the identification explained in Remark 5.3.

Remark 5.5. In Definition 5.4, for a deformation $\widetilde{D}$ of D that is trianguline, there may be several triangulations $\mathrm{Fil}^\bullet(\widetilde{D})$ on $\widetilde{D}$ that lift $\mathcal{F}^\bullet$. If the triangulation parameters $(\delta_1,\dots,\delta_n)$ of $\mathcal{F}^\bullet$ satisfy the condition that $\delta_i/\delta_j\notin\left\{x^{\mathbf{k}}\middle|\mathbf{k}\in\mathbb{N}^{\Sigma_K}\right\}$ for $1\le i<j\le n$, then Lemma 5.2(ii) applies to guarantee the uniqueness of the $\mathcal{F}^\bullet$-triangulation $\mathrm{Fil}^\bullet(\widetilde{D})$ on $\widetilde{D}$.

To verify that $\mathrm{Ext}^1_{\mathcal{F}^\bullet}(D,D)$ is an E-*subspace* of $\mathrm{Ext}^1(D,D)$, we recall the definition of Baer sum. Let B,C be two (φ,Γ_K)-modules over $\mathcal{R}_{K,E}$. For any two extensions in $\mathrm{Ext}^1(B,C)$,

$$0\to C\to D_1\to B\to 0,\quad 0\to C\to D_2\to B\to 0,$$

their Baer sum "D_1+D_2" is calculated by taking the direct sum

$$0\to C\oplus C\to D_1\oplus D_2\to B\oplus B\to 0,$$

pushing out via the sum $C\oplus C\xrightarrow{\Sigma}C$, and then pulling back via the diagonal $B\xrightarrow{\Delta}B\oplus B$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C\oplus C & \longrightarrow & D_1\oplus D_2 & \longrightarrow & B\oplus B & \longrightarrow & 0\\
 & & \downarrow\Sigma & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & C\sqcup_{C\oplus C}(D_1\oplus D_2) & \longrightarrow & B\oplus B & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow\Delta & & \\
0 & \longrightarrow & C & \longrightarrow & D_1+D_2 & \longrightarrow & B & \longrightarrow & 0
\end{array}\tag{5.1.2}$$

Moreover, if there are further extensions of (φ, Γ_K)-modules

$$0 \to C' \to C \to C'' \to 0, \quad 0 \to D'_i \to D_i \to D''_i \to 0, \quad 0 \to B' \to B \to B'' \to 0,$$

for $i = 1, 2$ that induce 9-term commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & D'_i & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & D_i & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'' & \longrightarrow & D''_i & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{5.1.3}$$

for $i = 1, 2$, then we naturally have a complex

$$0 \to (D'_1 + D'_2) \to (D_1 + D_2) \to (D''_1 + D''_2) \to 0 \tag{5.1.4}$$

which is exact, since after taking pushout and pullback, we have a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & D'_1 + D'_2 & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & D_1 + D_2 & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'' & \longrightarrow & D''_1 + D''_2 & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and two exact columns. By the 9-lemma, the middle column (5.1.4) is exact.

Proposition 5.6. *For any triangulation $\mathcal{F}^\bullet$ on a (φ, Γ_K)-module D of rank n over $\mathcal{R}_{K,E}$, the subspace of $\mathcal{F}^\bullet$-trianguline deformations $\mathrm{Ext}^1_{\mathcal{F}^\bullet}(D, D)$ is an E-linear subspace of $\mathrm{Ext}^1(D, D)$.*

Proof. Let $\widetilde{D}_1$ and $\widetilde{D}_2$ be two deformations of D to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$ lying in the subset $\mathrm{Ext}^1_{\mathcal{F}^\bullet}(D, D)$, respectively equipped with $\mathcal{F}^\bullet$-triangulations $\mathrm{Fil}^\bullet(\widetilde{D}_1)$ and $\mathrm{Fil}^\bullet(\widetilde{D}_2)$. For each $1 \le j \le n$ and

$i=1,2$, we naturally have a commutative diagram similar to (5.1.3) by

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^j & \longrightarrow & \mathrm{Fil}^j(\widetilde{D}_i) & \longrightarrow & \mathcal{F}^j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & \widetilde{D}_i & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D/\mathcal{F}^j & \longrightarrow & \widetilde{D}_i/\mathrm{Fil}^j(\widetilde{D}_i) & \longrightarrow & D/\mathcal{F}^j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence, by the exactness of (5.1.4), we have exact sequences

$$0 \to \mathrm{Fil}^j(\widetilde{D}_1)+\mathrm{Fil}^j(\widetilde{D}_2) \to \widetilde{D}_1+\widetilde{D}_2 \to (\widetilde{D}_1/\mathrm{Fil}^j(\widetilde{D}_1))+(\widetilde{D}_2/\mathrm{Fil}^j(\widetilde{D}_2)) \to 0$$

for $1\le j\le n$, which shows that $\mathrm{Fil}^\bullet(\widetilde{D}_1)+\mathrm{Fil}^\bullet(\widetilde{D}_2)$ is an $\mathcal{F}^\bullet$-triangulation on $\widetilde{D}_1+\widetilde{D}_2$. Note that by applying Baer sum to the following commutative diagrams for $i=1,2$,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^{j-1} & \longrightarrow & \mathrm{Fil}^{j-1}(\widetilde{D}_i) & \longrightarrow & \mathcal{F}^{j-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^j & \longrightarrow & \mathrm{Fil}^j(\widetilde{D}_i) & \longrightarrow & \mathcal{F}^j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^j(\mathcal{F}^\bullet) & \longrightarrow & \mathrm{gr}^j(\mathrm{Fil}^\bullet(\widetilde{D}_i)) & \longrightarrow & \mathrm{gr}^j(\mathcal{F}^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we obtain that $\mathrm{gr}^j(\mathrm{Fil}^\bullet(\widetilde{D}_1)+\mathrm{Fil}^\bullet(\widetilde{D}_2)) = \mathrm{gr}^j(\mathrm{Fil}^\bullet(\widetilde{D}_1))+\mathrm{gr}^j(\mathrm{Fil}^\bullet(\widetilde{D}_2))$ for $1\le j\le n$. To show that $\mathrm{Ext}^1_{\mathcal{F}}(D,D)$ is stable under E, let $0\to D\to \widetilde{D}\to D\to 0$ be a deformation of D equipped with an $\mathcal{F}^\bullet$-triangulation $\mathrm{Fil}^\bullet(\widetilde{D})$. For $\alpha\in E$, the scalar multiple $\alpha[\widetilde{D}]$ is the extension obtained by pushing $0\to D\to\widetilde{D}\to D\to 0$ out along the map $D\xrightarrow{\alpha} D$. Then, $\alpha[\mathrm{Fil}^\bullet(\widetilde{D})]$ is an $\mathcal{F}^\bullet$-triangulation on $\alpha[\widetilde{D}]$. So, $\mathrm{Ext}^1_{\mathcal{F}^\bullet}(D,D)$ is an E-linear subspace of $\mathrm{Ext}^1(D,D)$. □

For (φ,Γ_K)-module D of rank 1, its deformation to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$ has the following description.

Proposition 5.7. *For any continuous character $\delta: K^\times\to E^\times$, we have a canonical E-linear isomorphism*

$$\mathrm{Hom}(K^\times,E)\xrightarrow{\sim}\mathrm{Ext}^1(\mathcal{R}_{K,E}(\delta),\mathcal{R}_{K,E}(\delta)),\quad \psi\mapsto\mathcal{R}(\delta(1+\psi\varepsilon)),$$

where $\mathrm{Hom}(K^\times, E)$ *denotes the vector space of continuous homomorphisms from* $K^\times$ *to* E.

Proof. Twisting by δ^{-1} gives an isomorphism $\mathrm{Ext}^1(\mathcal{R}_{K,E}(\delta), \mathcal{R}_{K,E}(\delta)) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}_{K,E}, \mathcal{R}_{K,E})$. So we may assume that $\delta = 1$. Explicitly, given a continuous character $\psi : K^\times \to E$, we have

$$(1 + \psi(a)\varepsilon)(1 + \psi(b)\varepsilon) = 1 + (\psi(a) + \psi(b))\varepsilon = 1 + \psi(ab)\varepsilon$$

for any $a, b \in K^\times$. Hence, $1 + \psi\varepsilon : K^\times \to (E[\varepsilon]/\varepsilon^2)^\times$ is a continuous character that lifts the trivial character 1, and $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}(1 + \psi\varepsilon)$ deforms $\mathcal{R}_{K,E}$. Conversely, given any deformation $\widetilde{D}$ of $\mathcal{R}_{K,E}$ to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$, by Theorem 2.5 there is a unique character $\widetilde{\delta} : K^\times \to (E[\varepsilon]/\varepsilon^2)^\times$ such that $(\widetilde{\delta} \mod \varepsilon) = 1$ and $\widetilde{D} \cong \mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}(\widetilde{\delta})$. So, $\widetilde{\delta} = 1 + \psi\varepsilon$ for a continuous map $\psi : K^\times \to E$, which is a homomorphism because $\widetilde{\delta}$ is. The map $\mathrm{Hom}(K^\times, E) \to \mathrm{Ext}^1(\mathcal{R}_{K,E}, \mathcal{R}_{K,E})$ is an E-linear map because it is the comparison isomorphism [KPX14, Proposition 2.3.7]

$$H^1_{\mathrm{cont}}(G_K, V) \underset{\sim}{\xrightarrow{D_{\mathrm{rig}}}} H^1_{\varphi,\Gamma_K}(D_{\mathrm{rig}}(V))$$

between continuous Galois cohomology and (φ, Γ_K)-cohomology for the trivial 1-dimensional representation $V := E$ of G_K and the associated (φ, Γ_K)-module $D_{\mathrm{rig}}(V) = \mathcal{R}_{K,E}$. Indeed, by local class field theory, we have isomorphisms

$$H^1_{\mathrm{cont}}(G_K, E) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{K^\times}, E) = \mathrm{Hom}(K^\times, E),$$

where a continuous character $\psi \in \mathrm{Hom}(K^\times, E)$ corresponds to the deformation $E(1 + \psi\varepsilon) \in \mathrm{Ext}^1_{G_K}(E, E) \cong H^1_{\mathrm{cont}}(G_K, E)$ of the trivial representation E. Under D_{rig}, this gives our map

$$\mathrm{Hom}_{\mathrm{cont}}(K^\times, E) \to \mathrm{Ext}^1(\mathcal{R}_{K,E}, \mathcal{R}_{K,E}), \quad \psi \mapsto \mathcal{R}_{K,E}(1 + \psi\varepsilon) = D_{\mathrm{rig}}(E(1 + \psi\varepsilon)). \qquad \square$$

5.2. **Deformations of locally analytic principal series.** We slightly generalize results on translation of locally analytic principal series in [JLS24] to Artinian coefficients. Let $A \in \mathcal{C}_E$ be a local Artinian E-algebra of maximal ideal $\mathfrak{m}_A$ and residue field $A/\mathfrak{m}_A \cong E$.

5.2.1. Let $\mathbf{G} \supset \mathbf{P} \supset \mathbf{B} \supset \mathbf{T}$ be a split connected reductive group over K together with a parabolic subgroup, a Borel subgroup and a K-split maximal split torus. Write $G := \mathbf{G}(K), P := \mathbf{P}(K), B := \mathbf{B}(K), T := \mathbf{T}(K)$ for their groups of K-points. Let $\mathcal{G} \supset \mathcal{P} \supset \mathcal{B} \supset \mathcal{T}$ be the Weil restrictions of scalars from K to $\mathbb{Q}_p$, i.e., $\mathcal{G} := \mathrm{Res}^K_{\mathbb{Q}_p}(\mathbf{G}), \mathcal{B} := \mathrm{Res}^K_{\mathbb{Q}_p}(\mathbf{B})$, etc. Then, $\mathcal{G}(\mathbb{Q}_p) = G, \mathcal{P}(\mathbb{Q}_p) = P, \mathcal{B}(\mathbb{Q}_p) = B, \mathcal{T}(\mathbb{Q}_p) = T$ are locally K-analytic groups, whose $\mathbb{Q}_p$-Lie algebras are naturally K-vector spaces that we denote by $\mathfrak{g}, \mathfrak{p}, \mathfrak{b}, \mathfrak{t}$ respectively. Denote their base changes from $\mathbb{Q}_p$ to E by $\mathfrak{g}_E, \mathfrak{p}_E, \mathfrak{b}_E, \mathfrak{t}_E$, and let $\mathfrak{t}^*_E$ be the *weight space*, which is the E-linear dual of $\mathfrak{t}_E$. Via the isomorphism $K \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_K} E$, we have $\mathfrak{g}_E = \prod_\sigma \mathfrak{g}_E \otimes_{K,\sigma} E$, $\mathfrak{b}_E = \prod_\sigma \mathfrak{b}_E \otimes_{K,\sigma} E$, etc. Then, $\mathfrak{g}_E$ is a split reductive Lie algebra over E with Cartan subalgebra $\mathfrak{t}_E$ because $\mathcal{G} \times_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_K} \mathbf{G} \times_{K,\sigma} E$ is a split connected reductive group over E of Lie algebra $\mathfrak{g}_E$ with split maximal torus $\mathcal{T} \times_{\mathbb{Q}_p} E$. Via the inclusion

$$\mathbf{G}(K) = \mathcal{G}(\mathbb{Q}_p) \subset \mathcal{G}(E) = (\mathcal{G} \times_{\mathbb{Q}_p} E)(E),$$

we can use the highest weight theory for the split E-group $\mathcal{G} \times_{\mathbb{Q}_p} E$: if $L(\lambda)$ is the irreducible algebraic $\mathcal{G} \times_{\mathbb{Q}_p} E$-representation of highest weight $\lambda = (\lambda_\sigma)_{\sigma \in \Sigma_K} \in \mathfrak{t}^*_E \cong \prod_\sigma \mathfrak{t}^* \otimes_{K,\sigma} E$, then its restriction to $G = \mathbf{G}(K)$ is still irreducible by Zariski density of $\mathcal{G}(\mathbb{Q}_p)$ in $\mathcal{G}(E)$, and its differential is the irreducible $U(\mathfrak{g}_E)$-module of highest weight λ, where $U(\mathfrak{g}_E)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_E$.

Let $D(G)$ be the locally $\mathbb{Q}_p$-analytic distribution algebra of G over E, which is defined to be the strong dual of the locally convex topological E-vector space $C^{\mathrm{la}}(G,E)$ of locally $\mathbb{Q}_p$-analytic E-valued functions on G. Similarly, we define $D(B), D(P)$ with $D(B) \subset D(P) \subset D(G)$. Let $D(\mathfrak{g}, P)$ be the smallest subring of $D(G)$ containing both $U(\mathfrak{g}_E)$ and $D(P)$, cf. [JLS24, §2.1.4]. We have a canonical embedding $U(\mathfrak{g}_E) \hookrightarrow D(G)$ such that the center $\mathfrak{z}_E := Z(U(\mathfrak{g}_E))$ embeds into the center $Z(D(G))$. We have the *translation functor* on $\mathfrak{z}$-finite $D(G)$-modules

$$T_\lambda^\mu : D(G)\text{-mod}_{\mathfrak{z}\text{-fin}} \to D(G)\text{-mod}_{\mathfrak{z}\text{-fin}}, \quad M \mapsto \mathrm{pr}_{|\mu|}(L(\overline{\nu}) \otimes_E \mathrm{pr}_{|\lambda|}(M))$$

whenever $\lambda, \mu \in \mathfrak{t}_E^*$ are such that $\mu - \lambda$ lifts to an $E^\times$-valued algebraic character of $\mathbf{T}$. Here, $\mathrm{pr}_{|\mu|}$ is the projection to the generalized eigenspace of $\mathfrak{z}_E$ for its eigencharacter χ_μ determined by the action of $\mathfrak{z}_E$ on the Verma module $U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} E_\mu$, and $\overline{\nu}$ is the unique dominant weight in the Weyl orbit of the integral weight $\nu := \mu - \lambda \in \mathfrak{t}_E^*$. For any $\lambda \in \mathfrak{t}_E^*$, let $\mathfrak{m}_{\chi_\lambda} := \ker(\chi_\lambda) \subset Z(U(\mathfrak{g}_E))$ be the maximal ideal given by the kernel of χ_λ. For a locally $\mathbb{Q}_p$-analytic representation V of G on topological E-vector space of compact type, let V_b' denote its strong dual, which is naturally a separately continuous $D(G)$-module. For a locally $\mathbb{Q}_p$-analytic representation V of P over E, let $\mathrm{Ind}_P^G(V)^{\mathrm{la}}$ be the locally $\mathbb{Q}_p$-analytic parabolic induction

$$\mathrm{Ind}_P^G(V)^{\mathrm{la}} = \{f : G \to V \text{ locally } \mathbb{Q}_p\text{-analytic} \,|\, f(hg) = h.f(g), \forall g \in G, \forall h \in P\}.$$

Remark 5.8. This is slightly more general than the setup of [JLS24], where the authors take a split connected reductive group $\mathbf{H}$ over F and consider locally F-analytic representations of $\mathbf{H}(F)$. We are taking $F = \mathbb{Q}_p$ and $\mathbf{H} = \mathrm{Res}_{\mathbb{Q}_p}^K(\mathbf{G})$ so that $\mathbf{H}(F) = G$, which may not be split over $\mathbb{Q}_p$. But since we always consider locally $\mathbb{Q}_p$-analytic representations on E-vector spaces, and $\mathbf{H} \times_{\mathbb{Q}_p} E \cong \prod_{\sigma \in \Sigma_K} \mathbf{G} \times_{K,\sigma} E$ is a split connected reductive group over E, the results in [JLS24] can be proved in this case using the same arguments. Cf. [Bre16, §2] and [Dinb, §1].

Example 5.9. Here we generalize [JLS24, Example 4.1.4, Example 4.1.7]. Let $\tau_A : P \to A^\times$ be a locally $\mathbb{Q}_p$-analytic[7] character. Let $d\tau_A : \mathfrak{p} \to A$ be the $\mathbb{Q}_p$-linear derivative of τ_A, which induces an E-linear map $d\tau_A : \mathfrak{p}_E \to A$ that defines a $U(\mathfrak{p}_E)$-module structure on A. More generally, the locally analytic function $\tau_A \in C^{\mathrm{la}}(P, A) \cong C^{\mathrm{la}}(P,E) \otimes_E A$ induces an E-linear map $\chi_{\tau_A} : D(P) = C^{\mathrm{la}}(P,E)_b' \to A$. One can verify that χ_{τ_A} is an E-algebra map as τ_A is a group homomorphism. Let M be a finitely generated A-module.

(i) We write M_{τ_A} for the P-representation given by the composite $P \xrightarrow{\tau_A} A^\times \to \mathrm{Aut}_E(M)$, which is also a $D(P)$-module by the composite $D(P) \xrightarrow{\chi_{\tau_A}} A \to \mathrm{End}_E(M)$. Let $(M_{\tau_A})^\vee := \mathrm{Hom}_E(M_{\tau_A}, E)$ be the E-linear dual on which P acts by the contragradient action.

(ii) We write $M_{d\tau_A}$ for the $U(\mathfrak{p}_E)$-module structure on M given by the composite $U(\mathfrak{p}_E) \hookrightarrow D(P) \xrightarrow{\chi_{\tau_A}} A \to \mathrm{End}_E(M)$.

(iii) The induction $U(\mathfrak{g}_E) \otimes_{U(\mathfrak{p}_E)} M_{d\tau_A}$ is a $U(\mathfrak{g}_E)$-module equipped with a locally $\mathbb{Q}_p$-analytic P-action given by the formula $h(u \otimes m) = \mathrm{Ad}(h)(u) \otimes \tau_A(h)m$ for $h \in P, u \in U(\mathfrak{g}_E)$ and $m \in M_{d\tau_A}$, where Ad denotes the adjoint action of G on $U(\mathfrak{g}_E)$. The actions of $U(\mathfrak{g}_E)$ and P are compatible in the sense that $U(\mathfrak{g}_E) \otimes_{U(\mathfrak{p}_E)} M_{d\tau_A}$ is an object of the category $\mathcal{O}^{P,\infty}$ [JLS24, §4.1.2], which is naturally a $D(\mathfrak{g}, P)$-module by [AS22, Lemma 7.4.2].

(iv) The same argument given in [JLS24, Example 4.1.7] shows that the inclusion $U(\mathfrak{g}_E) \hookrightarrow D(\mathfrak{g}, P)$ induces an A-linear isomorphism of $D(\mathfrak{g}, P)$-modules

$$U(\mathfrak{g}_E) \otimes_{U(\mathfrak{p}_E)} M_{d\tau_A} \xrightarrow{\sim} D(\mathfrak{g}, P) \otimes_{D(P)} M_{\tau_A},$$

[7] Equivalently, a continuous character $\tau_A : P \to A^\times$ by [Buz07, Proposition 8.3].

and hence an A-linear isomorphism of $D(G)$-modules

$$\check{\mathcal{F}}_P^G(U(\mathfrak{g}_E)\otimes_{U(\mathfrak{p}_E)} M_{d\tau_A}) := D(G)\otimes_{D(\mathfrak{g},P)}(U(\mathfrak{g}_E)\otimes_{U(\mathfrak{p}_E)} M_{d\tau_A}) \xrightarrow{\sim} D(G)\otimes_{D(P)} M_{\tau_A}.$$

Lemma 5.10. *For a finite A-module M and locally $\mathbb{Q}_p$-analytic character $\tau_A : P \to A^\times$, we have the following isomorphism of $D(G)$-modules*

$$D(G)\otimes_{D(P)}(M_{\tau_A})^\vee \xrightarrow{\sim} (\mathrm{Ind}_P^G(M_{\tau_A})^{\mathrm{la}})'_b, \quad (\mu\otimes\ell)\mapsto (f\mapsto \mu(\ell\circ f)) \tag{5.2.1}$$

for any distribution $\mu \in D(G)$, linear functional $\ell \in (M_{\tau_A})^\vee$, and function $f \in \mathrm{Ind}_P^G(M_{\tau_A})^{\mathrm{la}}$.

Proof. Since A is local Artinian and M is finitely generated over A, M is of finite length over A with successive quotients given by $A/\mathfrak{m}_A \cong E$. Thus, the submodule $M[\mathfrak{m}_A]$ is nonzero, and any nonzero $m \in M[\mathfrak{m}_A]$ generates a 1-dimensional E-vector space $Am = Em$ on which P acts by the mod-$\mathfrak{m}_A$ reduction $\tau := (\tau_A \mod A)$ of τ_A. Consider

$$0 \to E_\tau \to M_{\tau_A} \to (M/Am)_{\tau_A} \to 0,$$

which is an exact sequence of $D(P)$-modules.

(i) Applying the locally analytic parabolic induction and continuous dual, which are exact, we get an exact sequence

$$0 \to (\mathrm{Ind}_P^G((M/Am)_{\tau_A})^{\mathrm{la}})'_b \to (\mathrm{Ind}_P^G(M_{\tau_A})^{\mathrm{la}})'_b \to (\mathrm{Ind}_P^G(E_\tau)^{\mathrm{la}})'_b \to 0$$

of $D(G)$-modules.

(ii) Applying the exact E-linear dual and $D(G)\otimes_{D(P)}(-)$, we get a sequence

$$0 \to D(G)\otimes_{D(P)}((M/Am)_{\tau_A})^\vee \to D(G)\otimes_{D(P)}(M_{\tau_A})^\vee \to D(G)\otimes_{D(P)}(E_\tau)^\vee \to 0$$

which is exact, since $\mathrm{Tor}_1^{D(P)}(D(G),(M/Am)_{\tau_A})^\vee) = 0$ by [ST05, Lemma 6.3(i)], where the hypothesis (FIN) in *loc. cit.* is satisfied by $(M/Am)_{\tau_A})^\vee$, as $(E_\tau)^\vee \cong E_{\tau^{-1}}$ satisfies (FIN) by [ST05, Lemma 6.5], $(M/Am)_{\tau_A})^\vee$ is a successive extension of $(E_\tau)^\vee$ as $D(P)$-module, and (FIN) is stable under extensions by [Wei94, Horseshoe Lemma 2.2.8].

The natural map (5.2.1) defines a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(G)\otimes_{D(P)}((M/Am)_{\tau_A})^\vee & \longrightarrow & D(G)\otimes_{D(P)}(M_{\tau_A})^\vee & \longrightarrow & D(G)\otimes_{D(P)}(E_\tau)^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (5.2.1)} & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Ind}_P^G((M/Am)_{\tau_A})^{\mathrm{la}})'_b & \longrightarrow & (\mathrm{Ind}_P^G(M_{\tau_A})^{\mathrm{la}})'_b & \longrightarrow & (\mathrm{Ind}_P^G(E_\tau)^{\mathrm{la}})'_b & \longrightarrow & 0
\end{array}$$

The quotient M/Am has shorter length than M as A-module, hence we have by induction on $\dim_E(M)$ that (5.2.1) holds for $(M/Am)_{\tau_A}$. The validity of (5.2.1) when $\dim_E(M) = 1$ follows from [ST05, Lemma 6.1(iv)]. By the 5-lemma, (5.2.1) holds for M_{τ_A}. $\square$

Remark 5.11. Suppose the local Artinian E-algebra A is *Frobenius* [Wei94, Definition 4.2.5], i.e., there is an A-module isomorphism $A \xrightarrow{\sim} A^\vee = \mathrm{Hom}_E(A,E)$. For any finite free A-module, we have $(M_{\tau_A})^\vee \cong M_{\tau_A^{-1}}$ as P-modules. For example, $E[\varepsilon]/\varepsilon^2$ is a Frobenius E-algebra since

$$E[\varepsilon]/\varepsilon^2 \times E[\varepsilon]/\varepsilon^2 \to E, \quad (a+b\varepsilon, c+d\varepsilon)\mapsto ad+bc$$

is an $E[\varepsilon]/\varepsilon^2$-equivariant non-degenerate bilinear pairing, showing that $E[\varepsilon]/\varepsilon^2 \cong (E[\varepsilon]/\varepsilon^2)^\vee$ as $E[\varepsilon]/\varepsilon^2$-modules. We therefore have $(A_{\tau_A})^\vee \cong A_{\tau_A^{-1}}$ for $A = E[\varepsilon]/\varepsilon^2$.

Lemma 5.12. *Let $\lambda, \mu \in \mathfrak{t}_E^*$ be such that $\lambda - \mu$ is the weight of an algebraic character of T. Let L_P be the Levi subgroup of P, and $\mathrm{Rep}_E^{\mathrm{sm}}(L_P)^{s\text{-}adm}$ be the category of strongly admissible smooth representations of L_P. For any pair $(M, V) \in \mathcal{O}^{P,\infty} \times \mathrm{Rep}_E^{\mathrm{sm}}(L_P)^{s\text{-}adm}$ such that M is an A-module where A acts by $D(\mathfrak{g}, P)$-linear endomorphisms and $\check{\mathcal{F}}_P^G(M, V) := D(G) \otimes_{D(\mathfrak{g},P)} (M \otimes_E V_b')$, there is a canonical A-linear isomorphism*

$$T_\lambda^\mu(\check{\mathcal{F}}_P^G(M, V)) \xrightarrow{\sim} \check{\mathcal{F}}_P^G(T_\lambda^\mu(M), V))$$

of $D(G)$-modules that is natural in M and V.

Proof. The isomorphism comes from [JLS24, Theorem 4.1.12], and it is natural in $M \in \mathcal{O}^{P,\infty}$ and $V \in \mathrm{Rep}_E^{\mathrm{sm}}(L_P)^{\text{s-adm}}$. Since the A-actions on both sides are induced by that on M, which commutes with the action of $D(\mathfrak{g}, P)$, this isomorphism is A-linear by its naturality in M. □

Lemma 5.13. *Let $\widetilde{\lambda}_A, \widetilde{\mu}_A : T \to A^\times$ be locally $\mathbb{Q}_p$-analytic characters of T, let $\widetilde{\lambda}, \widetilde{\mu} : T \to E^\times$ be their reductions modulo $\mathfrak{m}_A$, with weights $\lambda := d\widetilde{\lambda}, \mu := d\widetilde{\mu} \in \mathfrak{t}_E^*$. Suppose that*

(i) λ and μ are both anti-dominant,
(ii) $\widetilde{\nu} := \widetilde{\mu}_A/\widetilde{\lambda}_A$ is an $E^\times$-valued algebraic character of T, and
(iii) the condition (4.2.8)[8] of [JLS24] is satisfied by λ and μ.

Then, for any $w \in W_{[\lambda]} = W_{[\mu]}$ and finitely generated A-module M, there exists an A-linear isomorphism of $D(G)$-modules

$$T_\lambda^\mu(D(G) \otimes_{D(B)} M_{w \cdot \widetilde{\lambda}_A}) \cong D(G) \otimes_{D(B)} M_{w \cdot \widetilde{\mu}_A}.$$

Here, the dot action of $w \in W$ on the locally $\mathbb{Q}_p$-analytic character $\widetilde{\lambda}_A : T \to A^\times$ is defined via the same formula given in [JLS24, §4.2.3], so that the mod-$\mathfrak{m}_A$ reduction of $d(w \cdot \widetilde{\lambda}_A)$ equals $d(w \cdot \widetilde{\lambda}) = w \cdot \lambda$, which is the usual dot action of $w \in W$ on the weight $\lambda \in \mathfrak{t}_E^*$.

Proof. By Example 5.9(iv) and Lemma 5.12 (here we take $V = E$, the trivial representation of $L_B = T$), it suffices to prove that we have an A-linear isomorphism of $D(\mathfrak{g}, B)$-modules

$$T_\lambda^\mu(U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{d(w \cdot \widetilde{\lambda}_A)}) \cong U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{d(w \cdot \widetilde{\mu}_A)}.$$

Let $\overline{\nu}$ be the dominant weight in the Weyl orbit of $\nu := \mu - \lambda$, and let $L(\overline{\nu})$ be the irreducible G-representation of highest weight $\overline{\nu}$. Recall that $T_\lambda^\mu(N) = \mathrm{pr}_{|\mu|}(L(\overline{\nu}) \otimes_E \mathrm{pr}_{|\lambda|} N)$. For $N_A := U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{d(w \cdot \widetilde{\lambda}_A)}$, it is a successive extension of $N_A \otimes_A E = U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{d(w \cdot \widetilde{\lambda})}$, which lies in $U(\mathfrak{g}_E)\text{-mod}_{|\lambda|}$. Since $U(\mathfrak{g}_E)\text{-mod}_{|\lambda|}$ is stable under extensions, we get $\mathrm{pr}_{|\lambda|}(N_A) = N_A$. Then, we filter $L(\overline{\nu})|_B$ by B-stable subspaces with 1-dimensional graded pieces on which B acts through algebraic characters $\widetilde{\xi}_i : T \to E^\times$ as in [JLS24, Proposition 4.2.10], where $\xi_i \in \mathfrak{t}_E^*$ runs through all the weights of $L(\overline{\nu})$ counting multiplicities. Tensoring this filtration with N_A, we get a filtration on $L(\overline{\nu}) \otimes_E N_A$ by $D(\mathfrak{g}, B)$-submodules with graded pieces $D(\mathfrak{g}, B) \otimes_{D(B)} (E_{\widetilde{\xi}_i} \otimes_E M_{w \cdot \widetilde{\lambda}_A}) = D(\mathfrak{g}, B) \otimes_{D(B)} (M_{\widetilde{\xi}_i (w \cdot \widetilde{\lambda}_A)}) \cong U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{\xi_i + d(w \cdot \widetilde{\lambda}_A)}$, by Example 5.9(iv). Since $\mathrm{pr}_{|\mu|}$ is exact, $T_\lambda^\mu(N_A)$ has a filtration with graded pieces $\mathrm{pr}_{|\mu|}(U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{\xi_i + d(w \cdot \widetilde{\lambda}_A)})$ appearing with multiplicity $\dim_E(L(\overline{\nu})_{\xi_i})$ for each ξ_i. Again because the mod-$\mathfrak{m}_A$ reduction of $U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{\xi_i + d(w \cdot \widetilde{\lambda}_A)}$, and hence $U(\mathfrak{g}_E) \otimes_{U(\mathfrak{b}_E)} M_{\xi_i + d(w \cdot \widetilde{\lambda}_A)}$ itself, lies in $U(\mathfrak{g}_E)\text{-mod}_{|\xi_i + w \cdot \lambda|}$, its projection to $U(\mathfrak{g}_E)\text{-mod}_{|\mu|}$ is nonzero if and only if $\xi_i + w \cdot \lambda$ lies in the orbit of μ under the dot-action of W. Under the assumed condition (4.2.8) of [JLS24], by [JLS24, Corollary 4.2.9],

[8] It states that "$\mu^\natural$ lies in the closure of the facet $\mathscr{F} \subset \mathscr{E}(\lambda)$ which contains $\lambda^\natural$."

$\xi_i + w\cdot\lambda$ lies in the orbit of μ under the dot-action of W if and only if $\xi_i = w(\nu)$, in which case we have multiplicity $\dim_E(L(\overline{\nu})_{w(\nu)}) = \dim_E(L(\overline{\nu})_{\overline{\nu}}) = 1$. Therefore,

$$\begin{aligned} T_\lambda^\mu(U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E)} M_{d(w\cdot\widetilde{\lambda}_A)}) &= \mathrm{pr}_{|\mu|}(U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E)} M_{w(\nu)+d(w\cdot\widetilde{\lambda}_A)}) \\ &= U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E)} M_{w(\nu)+d(w\cdot\widetilde{\lambda}_A)} \\ &\cong D(\mathfrak{g},B)\otimes_{D(B)} M_{w(\widetilde{\nu})(w\cdot\widetilde{\lambda}_A)} \\ &= D(\mathfrak{g},B)\otimes_{D(B)} M_{w\cdot\widetilde{\mu}_A} \\ &\cong U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E)} M_{d(w\cdot\widetilde{\mu}_A)}, \end{aligned}$$

is the desired isomorphism of $D(\mathfrak{g},B)$-module, which is A-linear as each step is A-linear. □

5.3. **Ding's correspondence and the proof of Theorem B.** Let $K/\mathbb{Q}_p$ be a finite extension. Following [Din25], we take $\mathbf{G} = \mathrm{GL}_{n,K}$ in the setup of §5.2, with $\mathbf{B}$ the upper triangular Borel subgroup containing $\mathbf{T}$ the diagonal torus in $\mathbf{G}$. Let $\mathbf{B}^-$ be the lower triangular Borel subgroup. Then $G = \mathbf{G}(K) = \mathrm{GL}_n(K) \supset B = \mathbf{B}(K) \supset T = \mathbf{T}(K)$ and $B^- = \mathbf{B}^-(K)$.

Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ be non-critical crystabelline of regular Sen weights $\mathbf{h} = (h_{1,\sigma} > \cdots > h_{n,\sigma})_{\sigma\in\Sigma_K}$. Let $\mathbf{k} = (k_{i,\sigma}) \in (\mathbb{N}^n)^{\Sigma_K}$ and let $p_{\mathbf{k}}(D)$ be obtained from D by $p_{\mathbf{k}} = \prod_{i,\sigma} p_{i,\sigma}^{k_{i,\sigma}}$ such that the resulting weights $\mathbf{h}' = (h'_{1,\sigma} > \cdots > h'_{n,\sigma})_{\sigma\in\Sigma_K}$ are still regular. In this subsection, we recall the constructions of $\pi_{\min}(D) \subset \pi_{\mathrm{fs}}(D)$ from [Din25] and prove Theorem B from §1.1.6.

5.3.1. We recall Ding's parameter map κ_w on trianguline deformation spaces of D to $E[\varepsilon]/\varepsilon^2$. For any refinement $w \in S_n$ of ϕ, let $\mathrm{Fil}_w^\bullet(D)$ be the corresponding non-critical triangulation of D whose triangulation parameters are $\delta_{w,i} = \phi_{w(i)}\prod_{\sigma\in\Sigma_K} x_\sigma^{h_{i,\sigma}}$ for $1 \le i \le n$. Let $\mathrm{Ext}^1_w(D,D) \subset \mathrm{Ext}^1(D,D)$ be the subspace of $\mathrm{Fil}^\bullet_w(D)$-trianguline deformations to $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}$, cf. Remark 5.3 and Definition 5.4(ii). Since ϕ is generic, we can apply Lemma 5.2(ii) to conclude that on any deformation $\widetilde{D} \in \mathrm{Ext}^1_w(D,D)$, there is a unique triangulation $\mathrm{Fil}^\bullet(\widetilde{D})$ that lifts $\mathrm{Fil}^\bullet_w(D)$. The graded pieces of $\mathrm{Fil}^\bullet(\widetilde{D})$ deform $\mathcal{R}_{K,E}(\delta_{w,i})$, and hence are of the form $\mathcal{R}_{K,E[\varepsilon]/\varepsilon^2}(\delta_{w,i}(1+\psi_i\varepsilon))$ for continuous characters $\psi_i \in \mathrm{Hom}(K^\times, E)$ by Proposition 5.7. The tuple $\psi := (\psi_1,\dots,\psi_n)$ defines a continuous character $\psi : T \to E$, and following [Din25, (2.12)] we define

$$\kappa_w : \mathrm{Ext}^1_w(D,D) \to \mathrm{Hom}(T,E), \quad \widetilde{D} \mapsto \psi,$$

which is an E-linear map by Proposition 5.7. By [Din25, Proposition 2.10, Lemma 2.11], κ_w is surjective, and the subspace $\ker(\kappa_w)$ of $\mathrm{Ext}^1(D,D)$ is independent of $w \in S_n$. We denote this common subspace by $\mathrm{Ext}^1_0(D,D)$. For any subspace $X \subset \mathrm{Ext}^1(D,D)$ containing $\mathrm{Ext}^1_0(D,D)$, we write $\overline{X} := X/\mathrm{Ext}^1_0(D,D)$ for the quotient. Thus, we get an isomorphism

$$\kappa_w : \overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow{\sim} \mathrm{Hom}(T,E).$$

The sum over all refinements generates the full deformation space: the diagram

$$\begin{array}{ccc} \bigoplus_{w\in S_n} \mathrm{Ext}^1_w(D,D) & \twoheadrightarrow & \mathrm{Ext}^1(D,D) \\ \downarrow & & \downarrow \\ \bigoplus_{w\in S_n} \overline{\mathrm{Ext}}^1_w(D,D) & \twoheadrightarrow & \overline{\mathrm{Ext}}^1(D,D) \end{array}$$

commutes with surjective rows and columns, by [Din25, Proposition 2.12]

Lemma 5.14. *The operation $p_{\mathbf{k}}$ induces an E-linear isomorphism*

$$p_{\mathbf{k}} : \mathrm{Ext}^1(D, D) \xrightarrow{\sim} \mathrm{Ext}^1(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D))$$

such that for every refinement w, it restricts to

$$p_{\mathbf{k}} : \mathrm{Ext}^1_w(D, D) \xrightarrow{\sim} \mathrm{Ext}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)).$$

Moreover, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Ext}^1_w(D, D) & \xrightarrow{\kappa_w} & \mathrm{Hom}(T, E) \\ \downarrow p_{\mathbf{k}} & & \| \mathrm{id} \\ \mathrm{Ext}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) & \xrightarrow{\kappa_w} & \mathrm{Hom}(T, E) \end{array}$$

In other words, the pullback $p_{\mathbf{k}}$ does not change the parameter ψ.

Proof. Under the identification of $\mathrm{Ext}^1(D, D)$ with deformations of D to $E[\varepsilon]/\varepsilon^2$ in Remark 5.3, since D is of regular integral weights $\mathbf{h}$ and $\mathrm{Sp}\, E$ is the only closed point of $\mathrm{Sp}(E[\varepsilon]/\varepsilon^2])$, for any $\widetilde{D} \in \mathrm{Ext}^1(D, D)$, the (φ, Γ_K)-module $p_{\mathbf{k}}(\widetilde{D})$ equals $f_{\mathbf{h},\mathbf{h}'}(\widetilde{D})$ for the change of weights map $f_{\mathbf{h},\mathbf{h}'} : (\mathfrak{X}_n)^\wedge_{\mathbf{h}} \to (\mathfrak{X}_n)^\wedge_{\mathbf{h}'}$ of [Wu, §1.3]. To verify the properties, by induction we may assume $p_{\mathbf{k}} = p_{i,\sigma}$ for some i and σ, as in Theorem 4.10(i). By Theorem 4.10(iii), $p_{i,\sigma}$ is bijective. As for E-linearity, for $\widetilde{D} \in \mathrm{Ext}^1(D, D)$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & \widetilde{D} & \longrightarrow & D & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & p_{i,\sigma}(D) & \longrightarrow & p_{i,\sigma}(\widetilde{D}) & \longrightarrow & p_{i,\sigma}(D) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are inclusions. For $a \in E$, applying pushout by multiplication by a on the subobjects, we see that $p_{i,\sigma}(a[\widetilde{D}]) = a[p_{i,\sigma}(\widetilde{D})]$. As for additivity, we need to show that for any $\widetilde{D}_1, \widetilde{D}_2 \in \mathrm{Ext}^1(D, D)$ of the form

$$0 \to D \xrightarrow{i_1} \widetilde{D}_1 \xrightarrow{\pi_1} D \to 0, \quad 0 \to D \xrightarrow{i_2} \widetilde{D}_2 \xrightarrow{\pi_2} D \to 0$$

the Baer sums satisfy

$$p_{i,\sigma}(\widetilde{D}_1 + \widetilde{D}_2) = p_{i,\sigma}(\widetilde{D}_1) + p_{i,\sigma}(\widetilde{D}_2).$$

Since both sides are (φ, Γ_K)-submodules of the Baer sum $\widetilde{D}_1 + \widetilde{D}_2$ containing $t_\sigma(\widetilde{D}_1 + \widetilde{D}_2)$, by Theorem 4.10(i) it is enough to show that their Sen polynomials are equal. For $\widetilde{D} \in \mathrm{Ext}^1(D, D)$, its Sen polynomial $P_{\mathrm{Sen},\widetilde{D}}(T) \in (E[\varepsilon]/\varepsilon^2)[T]$ reduces modulo ε to the Sen polynomial $P_{\mathrm{Sen},D}(T) = \prod_{\tau\in\Sigma_K} \prod_{j=1}^n (T - h_{j,\tau}) \in \prod_{\tau\in\Sigma_K} E[T] \cong (K \otimes_{\mathbb{Q}_p} E)[T]$ of D. Since $E[\varepsilon]/\varepsilon^2$ is (ε)-adically complete, Hensel's lemma applies to show that

$$P_{\mathrm{Sen},\widetilde{D}}(T) = \prod_{\tau\in\Sigma_K} \prod_{j=1}^n (T - (h_j + a_{j,\tau}\varepsilon))$$

for some $a_{j,\tau} \in E$. Suppose for $l = 1, 2$, $P_{\mathrm{Sen},\widetilde{D}_l}(T) = \prod_{\tau\in\Sigma_K} \prod_{j=1}^n (T - (h_j + a^{(l)}_{j,\tau}\varepsilon))$ for some $a^{(l)}_{j,\tau} \in E$. We show that $P_{\mathrm{Sen},\widetilde{D}_1+\widetilde{D}_2}(T) = \prod_\tau \prod_j (T - (h_j + (a^{(1)}_{j,\tau} + a^{(2)}_{j,\tau})\varepsilon))$, from which it would follow that $p_{i,\sigma}(\widetilde{D}_1 + \widetilde{D}_2) = p_{i,\sigma}(\widetilde{D}_1) + p_{i,\sigma}(\widetilde{D}_2)$. Since $D_{\mathrm{Sen},\tau}$ is exact, for $l = 1, 2$, we can write $D_{\mathrm{Sen},\tau}(\widetilde{D}_l)=\varepsilon D_{\mathrm{Sen},\tau}(D)\oplus D_{\mathrm{Sen},\tau}(D)$ as free $K_\infty\otimes_{K,\tau}(E[\varepsilon]/\varepsilon^2)$-module of rank n. Let $\{e_{1,\tau},\ldots,e_{n,\tau}\}$ be an eigenbasis of the τ-Sen operator Θ_τ on $D_{\mathrm{Sen},\tau}(D)$. Then, $\{e_{1,\tau},\ldots,e_{n,\tau}\}$ is a $K_\infty\otimes_{K,\tau}(E[\varepsilon]/\varepsilon^2)$-basis of $D_{\mathrm{Sen},\tau}(\widetilde{D}_l)$ on which the τ-Sen operator $\widetilde{\Theta}_\tau$ acts by $\widetilde{\Theta}_\tau(e_{j,\tau})=(h_{j,\tau}+a^{(l)}_{j,\tau}\varepsilon)e_{j,\tau}$. The τ-Sen module of $\widetilde{D}_1+\widetilde{D}_2$ is a Baer sum of τ-Sen modules:

$$\begin{aligned}&D_{\mathrm{Sen},\tau}(\widetilde{D}_1+\widetilde{D}_2)=D_{\mathrm{Sen},\tau}(\widetilde{D}_1)+D_{\mathrm{Sen},\tau}(\widetilde{D}_2)\\&=\left\{(x_1,x_2)\in D_{\mathrm{Sen},\tau}(\widetilde{D}_1)\oplus D_{\mathrm{Sen},\tau}(\widetilde{D}_2)\middle|\pi_1(x_1)=\pi_2(x_2)\right\}/\left\{\varepsilon r,-\varepsilon r|r\in D_{\mathrm{Sen},\tau}(D)\right\},\end{aligned}$$

which has a $K_\infty\otimes_{K,\tau}(E[\varepsilon]/\varepsilon^2)$-basis represented by the classes of $\{(e_{j,\tau},e_{j,\tau})|1\le j\le n\}$ on which the τ-Sen operator $\widetilde{\Theta}_\tau$ acts by

$$\begin{aligned}\widetilde{\Theta}_{\mathrm{Sen},\tau}([e_{j,\tau},e_{j,\tau}])&=[(h_{j,\tau}+a^{(1)}_{j,\tau}\varepsilon)e_{j,\tau},(h_{j,\tau}+a^{(2)}_{j,\tau}\varepsilon)e_{j,\tau}]\\&=[(h_{j,\tau}+(a^{(1)}_{j,\tau}+a^{(2)}_{j,\tau})\varepsilon)e_{j,\tau},h_{j,\tau}e_{j,\tau}]\\&=(h_j+(a^{(1)}_{j,\tau}+a^{(2)}_{j,\tau})\varepsilon)[e_{j,\tau},e_{j,\tau}].\end{aligned}$$

Hence, $P_{\mathrm{Sen},\widetilde{D}_1+\widetilde{D}_2}(T)=\prod_\tau\prod_j(T-(h_j+(a^{(1)}_{j,\tau}+a^{(2)}_{j,\tau})\varepsilon))$, and the isomorphism

$$p_{i,\sigma}:\mathrm{Ext}^1(D,D)\xrightarrow{\sim}\mathrm{Ext}^1(p_{i,\sigma}(D),p_{i,\sigma}(D))$$

is E-linear. For each trianguline deformation $\widetilde{D}\in\mathrm{Ext}^1_w(D,D)$, $p_{i,\sigma}(\widetilde{D})$ is given by a pullback using the unique lifting of $\mathrm{Fil}^\bullet_w(D)$ and belongs to $\mathrm{Ext}^1_w(p_{i,\sigma}(D),p_{i,\sigma}(D))$ by construction. Moreover, the resulting triangulation on $p_{i,\sigma}(\widetilde{D})$ has parameters $x_\sigma\delta_{w,j}(1+\psi_j\varepsilon)$, from which we see that $\kappa_w(p_{i,\sigma}(\widetilde{D}))=(\psi_1,\ldots,\psi_n)=\kappa_w(\widetilde{D})$. □

Consequently, $p_{\mathbf{k}}$ preserves the subspace $\mathrm{Ext}^1_0(D,D)$ and induces an isomorphism

$$p_{\mathbf{k}}:\overline{\mathrm{Ext}}^1(D,D)\xrightarrow{\sim}\overline{\mathrm{Ext}}^1(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)).$$

Thus we have a natural commutative diagram

(5.3.1)
$$\begin{array}{ccc}\bigoplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D) & \twoheadrightarrow & \overline{\mathrm{Ext}}^1(D,D)\\ \downarrow{\scriptstyle p_{\mathbf{k}}} & & \downarrow{\scriptstyle p_{\mathbf{k}}}\\ \bigoplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D)) & \twoheadrightarrow & \overline{\mathrm{Ext}}^1(p_{\mathbf{k}}(D),p_{\mathbf{k}}(D))\end{array}$$

5.3.2. *Translation.* We fix notations for weights and translation functor on the automorphic side. Recall that $G=\mathbf{G}(K)=\mathrm{GL}_n(K)\supset B=\mathbf{B}(K)\supset T=\mathbf{T}(K)$ and $B^-=\mathbf{B}^-(K)$. Let

$$\rho:=\left(\frac{n-1}{2},\frac{n-3}{2},\ldots,\frac{1-n}{2}\right)_{\sigma\in\Sigma_K}\in E^{\Sigma_K}$$

be the half-sum of positive roots of G with respect to B. Then, the half-sum of the positive roots with respect to B^- is $\rho^-:=-\rho$. The Weyl group of $\mathrm{Res}^K_{\mathbb{Q}_p}(\mathbf{G})$ is $W\cong S_n^{\Sigma_K}$, whose element $w=(w_\sigma)_\sigma$ acts on $\mathfrak{t}^*_E=\prod_{\sigma\in\Sigma_K}\mathfrak{t}^*\otimes_{K,\sigma}E$ componentwise. Let $\theta:=(0,-1,\ldots,1-n)_{\sigma\in\Sigma_K}\in\mathbb{Z}^{\Sigma_K}$, which differs from ρ by a central character $\frac{n-1}{2}(1,\ldots,1)$. So, we can compute the dot-action $w\cdot\lambda:=w(\lambda+\rho)-\rho=w(\lambda+\theta)-\theta$ of Weyl group on weights $\lambda\in\mathfrak{t}^*_E$ using either ρ or θ.

For locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_n(K)$ on compact type E-vector spaces (all representations later in this article are of compact type), [Dinb, Lemma 3.9] relates the translation functor on those V that admit an infinitesimal character of $\mathfrak{z}_E$ with the translation functor on the strong dual V'_b: the following diagram commutes for any integral weights $\mu, \lambda \in \mathfrak{t}_E^*$,

$$\begin{array}{ccc} D(G)\text{-mod}_{\mathfrak{z}_E=\chi_{\lambda^*}} & \xrightarrow{T_{\lambda^*}^{\mu^*}} & D(G)\text{-mod}_{\mathfrak{z}_E=\chi_{\mu^*}} \\ {\scriptstyle (-)'_b}\uparrow & & {\scriptstyle (-)'_b}\uparrow \\ \mathrm{Rep}^{\mathrm{la}}(G)_{\mathfrak{z}_E=\chi_\lambda} & \xrightarrow{T_\lambda^\mu} & \mathrm{Rep}^{\mathrm{la}}(G)_{\mathfrak{z}_E=\mu_\lambda} \end{array} \tag{5.3.2}$$

where the vertical maps are taking the strong dual, which induce an anti-equivalence of categories from locally analytic representations of G on compact type E-vector spaces to separately continuous $D(G)$-modules on nuclear Fréchet spaces, cf. [ST02, Cor. 3.3], and $\lambda^* = -w_0\lambda$ is the dual highest weight with $w_0 \in W$ the longest Weyl group element, cf. [Bou75, VIII, §7, n° 5, prop. 11]. We will use the same compatibility for objects with bounded generalized infinitesimal character. Namely, if a compact-type locally analytic representation V is killed by a power $\mathfrak{m}_{\chi_\lambda}^N$ of the maximal ideal corresponding to χ_λ, then its strong dual V'_b is killed by the corresponding power $\mathfrak{m}_{\chi_{-w_0\lambda}}^N$. Since strong duality and translation functors are exact on the relevant $\mathfrak{z}_E$-finite categories, [Dinb, Lemma 3.9] extends to this bounded generalized setting by dévissage on N. In the applications below, $N \leq 2$, because the principal series are deformations over $E[\varepsilon]/\varepsilon^2$.

Lemma 5.15. *Let $\lambda' := \mathbf{h}' - \theta$ and $\lambda := \mathbf{h} - \theta$. The translation functors $T_{\lambda^*}^{\lambda'^*}$ and $T_\lambda^{\lambda'}$ in (5.3.2) are equivalence of categories.*

Proof. It is enough to show that $T_{\lambda^*}^{\lambda'^*} : D(G)\text{-mod}_{|\lambda^*|} \longrightarrow D(G)\text{-mod}_{|\lambda'^*|}$ is an equivalence of categories, for which we use [JLS24, Theorem 1]. Note that $T_{\lambda^*}^{\lambda'^*} = T_{w_0\cdot\lambda^*}^{w_0\cdot\lambda'^*}$ because $w_0 \cdot \lambda'^* - w_0 \cdot \lambda^* = w_0(\lambda'^* - \lambda^*)$. Then, $w_0 \cdot \lambda'^* - w_0 \cdot \lambda^* = \lambda - \lambda'$ lifts to an algebraic character of $\mathbf{T}$, both $w_0 \cdot \lambda'^*$ and $w_0 \cdot \lambda^*$ are anti-dominant, and both of them have the same (trivial) stabilizer for the dot-action as $\mathbf{h}, \mathbf{h}'$ are regular. Hence, [JLS24, Theorem 1] applies to yield that

$$T_{w_0\cdot\lambda^*}^{w_0\cdot\lambda'^*} = T_{\lambda^*}^{\lambda'^*} : D(G)\text{-mod}_{|\lambda^*|} \longrightarrow D(G)\text{-mod}_{|\lambda'^*|}$$

is an equivalence of categories. ☐

5.3.3. *Principal series.* We recall the deformations to $E[\varepsilon]/\varepsilon^2$ on the locally analytic side considered in [Din25, §3.1.2]. For any $w \in S_n$, we define the locally analytic principal series

$$\mathrm{PS}(w(\phi), \mathbf{h}) := \mathrm{Ind}_{B^-}^G(w(\phi)\eta z^\lambda)^{\mathrm{la}},$$

where $z^\lambda : T \to E^\times, (z_1, \ldots, z_n) \mapsto \prod_{i=1}^n \prod_{\sigma\in\Sigma_K} \sigma(z_i)^{\lambda_{i,\sigma}}$ is the algebraic character of weight λ, and $\eta := 1 \boxtimes |\cdot|_K \boxtimes \cdots \boxtimes |\cdot|_K^{n-1} = |\cdot|_K^{-1} \circ \theta$ is a smooth character added for the purpose of normalization. By [Din25, Proposition 3.1(2)], the $\mathrm{GL}_n(K)$-socle of $\mathrm{PS}(w(\phi), \mathbf{h})$ is independent of the choice of $w \in S_n$, and the socle is locally algebraic, given by

$$\pi_{\mathrm{alg}}(\phi, \mathbf{h}) := \mathrm{Ind}_{B^-}^G(\phi \cdot \eta)^\infty \otimes_E L(\lambda)$$

where $\mathrm{Ind}_{B^-}^G(-)^\infty$ denotes the smooth parabolic induction, and $L(\lambda)$ is the algebraic representation of G of highest weight λ with respect to B. For $\psi \in \mathrm{Hom}(T, E)$, let $i_{w,\mathbf{h}}(\psi)$ be the locally analytic self-extension of $\mathrm{PS}(w(\psi), \mathbf{h})$ given by the locally analytic principal series

$$i_{w,\mathbf{h}}(\psi) := \mathrm{Ind}_{B^-}^G(w(\phi)\eta z^\lambda(1 + \psi\varepsilon))^{\mathrm{la}}$$

with coefficients in $E[\varepsilon]/\varepsilon^2$, cf. §5.2, whose strong dual is isomorphic to

$$D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{(w(\phi)\eta z^{\lambda}(1+\psi\varepsilon))^{-1}}$$

by Lemma 5.10 and Remark 5.11. From this tensor product description, we deduce that

$$i_{w,\mathbf{h}}:\mathrm{Hom}(T,E)\to \mathrm{Ext}^1_G(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h}))$$

is an E-linear map.

Proposition 5.16. *For any $w\in S_n$ and $\psi\in\mathrm{Hom}(T,E)$, we have $T_{\lambda}^{\lambda'}(i_{w,\mathbf{h}}(\psi))\cong i_{w,\mathbf{h}'}(\psi)$. That is, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}(T,E) & \xrightarrow{i_{w,\mathbf{h}}} & \mathrm{Ext}^1_G(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})) \\ \Big\| \mathrm{id} & & \Big\downarrow T_{\lambda}^{\lambda'} \\ \mathrm{Hom}(T,E) & \xrightarrow{i_{w,\mathbf{h}'}} & \mathrm{Ext}^1_G(\mathrm{PS}(w(\phi),\mathbf{h}'),\mathrm{PS}(w(\phi),\mathbf{h}')) \end{array} \tag{5.3.3}$$

Proof. By (5.3.2), it suffices to prove the statement for the strong duals as $D(G)$-modules:

$$T_{\lambda^*}^{\lambda'^*}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{(w(\phi)\eta z^{\lambda}(1+\psi\varepsilon))^{-1}}\right)\cong D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{(w(\phi)\eta z^{\lambda'}(1+\psi\varepsilon))^{-1}},$$

which we will reduce to Lemma 5.13.

For this purpose, it is necessary to translate the language from the upper triangular Borel B to lower triangular Borel B^-. For $w\in W$ and $\xi\in\mathfrak{t}_E^*$, the Weyl dot-action with respect to B^- is given by $w\bar{\cdot}\xi=w(\xi+\rho^-)-\rho^-$. By the independence of the twisted Harish-Chandra isomorphism of the choice of the system of positive roots [KV95, p. 290] and the definition of the twisted Harish-Chandra isomorphism, one verifies (using the positive roots with respect to B and the positive roots with respect to B^-) that for any weight $\xi\in\mathfrak{t}_E^*$, the eigencharacter χ_ξ of $\mathfrak{z}_E$ on the Verma module $U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E)}E_\xi$ coincides with the eigencharacter $\chi^-_{w_0\xi}$ of $\mathfrak{z}_E$ on the *opposite* Verma module $U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E^-)}E_{w_0\xi}$. For any $\xi_1,\xi_2\in\mathfrak{t}_E^*$, let $\overline{T}_{\xi_1}^{\xi_2}$ denote the translation functor with respect to B^-. Then, we have $\overline{T}_{\xi_1}^{\xi_2}=T_{w_0\xi_1}^{w_0\xi_2}$.

Therefore, we have $T_{\lambda^*}^{\lambda'^*}=T_{-w_0\lambda}^{-w_0\lambda'}=\overline{T}_{-\lambda}^{-\lambda'}=\overline{T}_{w_0\bar{\cdot}(-\lambda)}^{w_0\bar{\cdot}(-\lambda')}$ and we need to show that

$$\begin{aligned}&\overline{T}_{w_0\bar{\cdot}(-\lambda)}^{w_0\bar{\cdot}(-\lambda')}\left(D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{w_0\bar{\cdot}w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda}(1-\psi\varepsilon))}\right)\\ &\cong D(G)\otimes_{D(B^-)}(E[\varepsilon]/\varepsilon^2)_{w_0\bar{\cdot}w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda'}(1-\psi\varepsilon))},\end{aligned}$$

which will follow from Lemma 5.13 by taking $A=E[\varepsilon]/\varepsilon^2$, $\widetilde{\lambda}_A=w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda}(1-\psi\varepsilon))$, $\widetilde{\mu}_A=w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda'}(1-\psi\varepsilon))$, and $w=w_0\in W_{[d\widetilde{\lambda}_A \bmod \varepsilon]}=W_{[d\widetilde{\mu}_A \bmod \varepsilon]}=W$. It remains to check that the hypotheses of Lemma 5.13 are satisfied by the above choices. For (i), since $\mathbf{h}$ and $\mathbf{h}'$ are regular, both $w_0\bar{\cdot}(-\lambda)=-w_0(\mathbf{h})+\theta$ and $w_0\bar{\cdot}(-\lambda')=-w_0(\mathbf{h}')+\theta$ are anti-dominant with respect to B^-. For (ii), the ratio

$$\begin{aligned}\widetilde{\mu}_A/\widetilde{\lambda}_A&=\frac{w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda'}(1-\psi\varepsilon))}{w_0\bar{\cdot}(w(\phi)^{-1}\eta^{-1}z^{-\lambda}(1-\psi\varepsilon))}\\ &=w_0(z^{\lambda-\lambda'})=z^{w_0(\lambda-\lambda')}\end{aligned}$$

is an $E^\times$-valued algebraic character of $\mathbf{T}(K)$. For (iii), both $(w_0\bar{\cdot}(-\lambda))^\natural$ and $(w_0\bar{\cdot}(-\lambda'))^\natural$ lie in the same open B^--anti-dominant chamber in the Euclidean space $\mathbb{R}\otimes_{\mathbb{Z}}\Phi$, for Φ the root system of $(\mathfrak{g}_E,\mathfrak{t}_E)$, cf. [JLS24, §4.2.3], because $\mathbf{h}$ and $\mathbf{h}'$ are regular. Hence, Lemma 5.13 applies. ☐

Recall from [Din25, Proposition 3.1] that the irreducible components of $\mathrm{PS}(w(\phi), \mathbf{h})$ are

$$\mathcal{F}^G_{B^-}(L^-(-u\cdot\lambda), w(\phi)\eta) = \big(D(G)\otimes_{D(\mathfrak{g},B^-)}(L^-(-u\cdot\lambda)\otimes_E (w(\phi)\eta)^{-1})\big)'_b$$

cf. [JLS24, Remark 4.1.11], for $w\in S_n$ and $u=(u_\sigma)_\sigma \in W = S_n^{\Sigma_K}$, where $L^-(-u\cdot\lambda)$ is the unique simple quotient of $U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E^-)} E_{-u\cdot\lambda}$, [Din25, §3.1.1]. We denote these representations by $\mathscr{C}^{(\phi,\mathbf{h})}(w,u) := \mathcal{F}^G_{B^-}(L^-(-u\cdot\lambda), w(\phi)\eta)$ to emphasize their dependence on ϕ and the weight $\mathbf{h}$. By Lemma 5.15 and [JLS24, Proposition 4.2.10], we have

$$T_\lambda^{\lambda'}(\mathscr{C}^{(\phi,\mathbf{h})}(w,u)) \cong \mathscr{C}^{(\phi,\mathbf{h}')}(w,u) \tag{5.3.4}$$

since $L^-(-u\cdot\lambda)$ is the unique simple quotient of $U(\mathfrak{g}_E)\otimes_{U(\mathfrak{b}_E^-)} E_{-u\cdot\lambda}$ and $T_\lambda^{\lambda'}$ is an equivalence of categories. Then, following [Din25, (3.2)], we let $\mathrm{PS}_1(w(\phi),\mathbf{h})$ be the unique subrepresentation of $\mathrm{PS}(w(\phi),\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h}) = \mathscr{C}^{(\phi,\mathbf{h})}(w,\mathrm{id})$ and cosocle $\bigoplus_{\substack{1\le i\le n-1\\ \sigma\in\Sigma_K}}\mathscr{C}^{(\phi,\mathbf{h})}(w,s_{i,\sigma})$, where $s_{i,\sigma}$ denotes the element of $W=S_n^{\Sigma_K}$ that is the simple reflection s_i at the σ-component and trivial elsewhere. By its uniqueness, (5.3.4), Lemma 5.15 and Proposition 5.16, we deduce

$$T_\lambda^{\lambda'}(\mathrm{PS}_1(w(\phi),\mathbf{h})) \cong \mathrm{PS}_1(w(\phi),\mathbf{h}'). \tag{5.3.5}$$

By [Din25, (3.3)], the amalgamated sum $\bigoplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\phi,\mathbf{h})}\mathrm{PS}_1(w(\phi),\mathbf{h})$ has a unique subquotient of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$, and this subquotient is denoted $\pi_1(\phi,\mathbf{h})$. By its uniqueness, (5.3.4), (5.3.5) and Lemma 5.15, we deduce that

$$T_\lambda^{\lambda'}(\pi_1(\phi,\mathbf{h})) \cong \pi_1(\phi,\mathbf{h}'). \tag{5.3.6}$$

After [Din25, Remark 3.9], we denote by $\pi(\phi,\mathbf{h})$ the unique quotient of $\bigoplus^{w\in S_n}_{\pi_{\mathrm{alg}}(\phi,\mathbf{h})}\mathrm{PS}_1(w(\phi),\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. We similarly deduce that

$$T_\lambda^{\lambda'}(\pi(\phi,\mathbf{h})) \cong \pi(\phi,\mathbf{h}'). \tag{5.3.7}$$

Following [Din25, (3.11)], for each $w\in S_n$ we define

$$\zeta_{w,\mathbf{h},1} : \mathrm{Hom}(T,E) \to \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})),$$

firstly sending $\psi\in\mathrm{Hom}(T,E)$ to $i_{w,\mathbf{h}}(\psi)\in\mathrm{Ext}^1_G(\mathrm{PS}(w(\phi),\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h}))$, and then pulling it back along the embedding $\pi_{\mathrm{alg}}(\phi,\mathbf{h})\hookrightarrow \mathrm{PS}_1(w(\phi),\mathbf{h})$ to

$$\mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h})) \cong \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h})),$$

where the isomorphism is induced by pushout along $\mathrm{PS}_1(w(\phi),\mathbf{h})\hookrightarrow\mathrm{PS}(w(\phi),\mathbf{h}))$, and finally pushing it forward to $\mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ along $\mathrm{PS}_1(w(\phi),\mathbf{h}))\hookrightarrow\pi_1(\phi,\mathbf{h})$. The same construction, using $\pi(\phi,\mathbf{h})$ instead of $\pi_1(\phi,\mathbf{h})$, gives the version

$$\zeta_{w,\mathbf{h}} : \mathrm{Hom}(T,E) \to \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})).$$

By [Din25, Remark 3.9], the pushforward along $\pi_1(\phi,\mathbf{h})\hookrightarrow\pi(\phi,\mathbf{h})$ is an isomorphism

$$\mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \cong \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})).$$

By [Din25, (3.17)], both $\zeta_{w,\mathbf{h},1}$ and $\zeta_{w,\mathbf{h}}$ are E-linear injections, and we denote their images by

$$\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \cong \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})).$$

Since $T_\lambda^{\lambda'}$ is an equivalence of categories, it commutes with pullbacks and pushforwards. Thus, together with Lemma 5.14 and Proposition 5.16, we deduce the following corollary.

Corollary 5.17. *(i) For any $w \in S_n$ and $\bullet = 1$ or empty, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(T, E) & \xhookrightarrow{\zeta_{w,\mathbf{h},\bullet}} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) \\ \Big\| \mathrm{id} & & \Big\downarrow T_\lambda^{\lambda'} \\ \mathrm{Hom}(T, E) & \xhookrightarrow{\zeta_{w,\mathbf{h}',\bullet}} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}')) \end{array}$$

(ii) For any $w \in S_n$ and $\bullet = 1$ or empty, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_w(D, D) & \xhookrightarrow{\zeta_{w,\mathbf{h},\bullet} \circ \kappa_w} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) \\ \Big\downarrow p_{\mathbf{k}} & & \Big\downarrow T_\lambda^{\lambda'} \\ \overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) & \xhookrightarrow{\zeta_{w,\mathbf{h}',\bullet} \circ \kappa_w} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}')) \end{array}$$

(iii) For $\bullet = 1$ or empty, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(D, D) & \xrightarrow{p_{\mathbf{k}}} & \bigoplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) \\ \Big\downarrow \zeta_{w,\mathbf{h},\bullet} \circ \kappa_w & & \Big\downarrow \zeta_{w,\mathbf{h}',\bullet} \circ \kappa_w \\ \bigoplus_{w \in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) & \xrightarrow{T_\lambda^{\lambda'}} & \bigoplus_{w \in S_n} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}')) \\ \Big\downarrow \mathrm{sum} & & \Big\downarrow \mathrm{sum} \\ \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) & \xrightarrow{T_\lambda^{\lambda'}} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}')) \end{array}$$

where the amalgamation maps "sum" are surjective by [Din25, Proposition 3.8(2)].

5.3.4. *Universal extension.* We discuss the notion of "universal representation" in general. Let E be a field, and let $\mathcal{D}$ be an E-linear abelian category with objects A and B. Then, to any finite dimensional E-linear subspace W of $\mathrm{Ext}^1_{\mathcal{D}}(A, B)$, one can attach a "universal extension" $\mathcal{E}^{W\text{-univ}}$ of $A \otimes_E W$ ($\cong A^{\dim_E W}$) by B, satisfying the property that for any extension

$$0 \to B \to \mathcal{E}_e \to A \to 0$$

with corresponding class $e \in W \subset \mathrm{Ext}^1_{\mathcal{D}}(A, B)$, the map $\alpha_e : A \to A \otimes_E W, a \mapsto a \otimes e$ induces

$$\alpha_e^* : \mathrm{Ext}^1_{\mathcal{D}}(A \otimes_E W, B) \to \mathrm{Ext}^1_{\mathcal{D}}(A, B)$$

such that the pullback $\alpha_e^*(\mathcal{E}^{W\text{-univ}})$ is precisely $\mathcal{E}_e$.

Proposition 5.18. *(i) Abstractly, this extension $\mathcal{E}^{W\text{-univ}}$ is the image of the inclusion map*

$$i_W : W \hookrightarrow \mathrm{Ext}^1_{\mathcal{D}}(A, B)$$

under the canonical isomorphisms

$$i_W \in \mathrm{Hom}_E(W, \mathrm{Ext}^1_{\mathcal{D}}(A, B)) \underset{\mathrm{can}}{\xleftarrow{\sim}} \mathrm{Ext}^1_{\mathcal{D}}(A, B) \otimes_E W^\vee \underset{\mathrm{can}}{\xrightarrow{\sim}} \mathrm{Ext}^1_{\mathcal{D}}(A \otimes_E W, B) \ni \mathcal{E}^{W\text{-univ}}$$

where the first map is a canonical isomorphism for finite dimensional W, and the second map is defined as follows: given $\mathcal{E}_e \in \mathrm{Ext}^1_{\mathcal{D}}(A, B)$ and a linear functional $f \in W^\vee$, we pullback $\mathcal{E}_e$ via $\beta_f : A \otimes W \to A, a \otimes w \mapsto f(w)a$ to $\beta_f^*(\mathcal{E}_e) \in \mathrm{Ext}^1_{\mathcal{D}}(A, B)$, which defines

$$\mathrm{Ext}^1_{\mathcal{D}}(A, B) \otimes_E W^\vee \underset{\mathrm{can}}{\xrightarrow{\sim}} \mathrm{Ext}^1_{\mathcal{D}}(A \otimes_E W, B), \quad \mathcal{E}_e \otimes f \mapsto \beta_f^*(\mathcal{E}_e)$$

with its inverse given by

$$\mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\xrightarrow[\mathrm{can}]{\sim}\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)),\quad \mathcal{E}\mapsto(e\mapsto\alpha_e^*(\mathcal{E})).$$

(ii) *Concretely, choose any E-basis $\{e_1,\dots,e_d\}$ of W, corresponding to extensions $\mathcal{E}_1,\dots,\mathcal{E}_d\in\mathrm{Ext}^1_{\mathcal{D}}(A,B)$. Form the pushout of $\mathcal{E}_1,\dots,\mathcal{E}_d$ over the common object B, denoted $\bigoplus_B^{1\le i\le d}\mathcal{E}_i$:*

$$\left(0\to B\to\bigoplus_B^{1\le i\le d}\mathcal{E}_i\to\bigoplus_{i=1}^d Ae_i\cong A\otimes_E W\to 0\right)\in\mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B).$$

Then, $\bigoplus_B^{1\le i\le d}\mathcal{E}_i\cong\mathcal{E}^{W\text{-univ}}$ is the universal extension associated to W.

Proof. For any $e\in W$, the inclusion $i_e:E\hookrightarrow W,1\mapsto e$ induces a commutative diagram:

$$\begin{array}{ccccc}\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\\ \downarrow{\scriptstyle i_e^*} & & \downarrow{\scriptstyle \mathrm{id}\otimes i_e^\vee} & & \downarrow{\scriptstyle \alpha_e^*}\\ \mathrm{Hom}_E(E,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E E^\vee & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E E,B)\end{array}$$

Keeping track of the element $i_W\in\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B))$ proves (i).

As for the second statement, we have a commutative diagram of E-linear isomorphisms

$$\begin{array}{ccccc}\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E W^\vee & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E W,B)\\ \simeq\downarrow{\scriptstyle \oplus_{i=1}^d i_{e_i}^*} & & \simeq\downarrow{\scriptstyle \oplus_{i=1}^d(\mathrm{id}\otimes i_{e_i}^\vee)} & & \simeq\downarrow{\scriptstyle \oplus_{i=1}^d\alpha_{e_i}^*}\\ \bigoplus_{i=1}^d\mathrm{Hom}_E(E,\mathrm{Ext}^1_{\mathcal{D}}(A,B)) & \xrightarrow{\sim} & \bigoplus_{i=1}^d\mathrm{Ext}^1_{\mathcal{D}}(A,B)\otimes_E E^\vee & \xrightarrow{\sim} & \bigoplus_{i=1}^d\mathrm{Ext}^1_{\mathcal{D}}(A\otimes_E E,B)\end{array}$$

for the chosen basis $\{e_1,\dots,e_d\}$ of W. Recall that the pushout $\bigoplus_B^{1\le i\le d}\mathcal{E}_i$ sitting in

$$0\to B\to\bigoplus_B^{1\le i\le d}\mathcal{E}_i\to\bigoplus_{i=1}^d Ae_i\cong A\otimes_E W\to 0$$

is by construction such that $\alpha_{e_j}^*\left(\bigoplus_B^{1\le i\le d}\mathcal{E}_i\right)=\mathcal{E}_j$ for each $1\le j\le d$. Hence, it corresponds to $i_W\in\mathrm{Hom}_E(W,\mathrm{Ext}^1_{\mathcal{D}}(A,B))$, and it equals the universal extension attached to W by (i). ☐

5.3.5. *Ding's correspondence and the proof of Theorem B.* We now recall Ding's construction of locally $\mathbb{Q}_p$-analytic $\mathrm{GL}_n(K)$-representations $\pi_{\min}(D)$ and $\pi_{\mathrm{fs}}(D)$ attached to $D\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$, and then give a proof of Theorem B. By [Din25, Theorem 3.21], there is a unique E-linear map

$$t_D:\mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1(D,D)$$

characterized by the property that the following diagram is commutative

$$\begin{array}{ccc}\bigoplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(D,D) & \xrightarrow[\sim]{\zeta_{w,\mathbf{h},1}\circ\kappa_w} & \bigoplus_{w\in S_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\\ \downarrow\!\!\downarrow{\scriptstyle \mathrm{sum}} & & \downarrow\!\!\downarrow{\scriptstyle \mathrm{sum}}\\ \overline{\mathrm{Ext}}^1(D,D) & \xleftarrow[t_D]{} & \mathrm{Ext}^1_G(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\end{array}\tag{5.3.8}$$

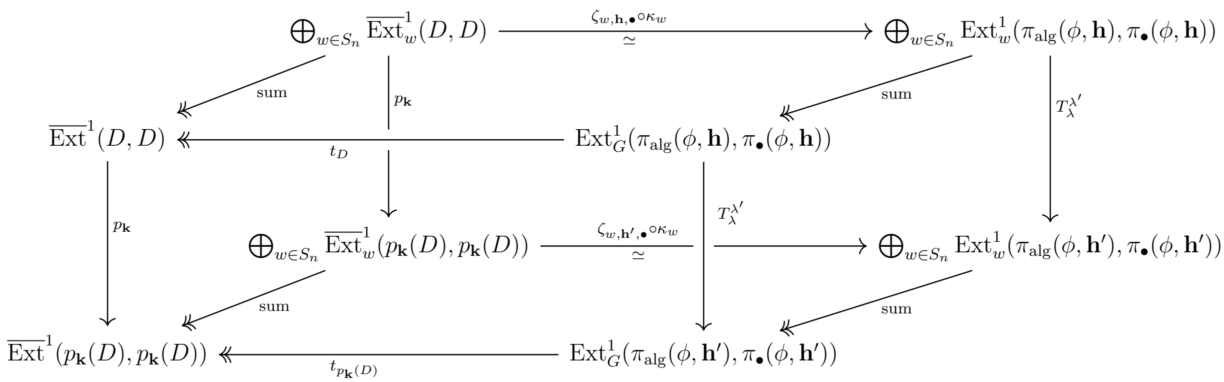
The same map t_D is used when $\pi_1(\phi,\mathbf{h})$ is replaced by $\pi(\phi,\mathbf{h})$, given [Din25, (3.16)]. In [Din25, below Lemma 3.23], Ding defines $\pi_{\min}(D)$ as the universal extension $\mathcal{E}^{\ker(t_D)\text{-univ}}$ of $\ker(t_D)\otimes_E$

$\pi_{\rm alg}(\phi, \mathbf{h})$ by $\pi_1(\phi, \mathbf{h})$ and defines $\pi_{\rm fs}(D)$ as the universal extension of $\ker(t_D) \otimes_E \pi_{\rm alg}(\phi, \mathbf{h})$ by $\pi(\phi, \mathbf{h})$ in the sense of §5.3.4. Then, $\pi_{\rm fs}(D) \cong \pi_{\rm min}(D) \oplus_{\pi_1(\phi,\mathbf{h})} \pi(\phi, \mathbf{h})$ by [Din25, (3.24)].

Proposition 5.19. *For $\bullet = 1$ or empty, we have a commutative diagram*

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1(D, D) & \xleftarrow{\ t_D\ } & \mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) \\ \downarrow{\scriptstyle p_{\mathbf{k}}} & & \downarrow{\scriptstyle T_\lambda^{\lambda'}} \\ \overline{\mathrm{Ext}}^1(p_{\mathbf{k}}(D), p_{\mathbf{k}}(D)) & \xleftarrow{\ t_{p_{\mathbf{k}}(D)}\ } & \mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}')) \end{array}$$

Proof. Consider the following cube whose "front" face is the square in the statement



All other five faces commute either by Corollary 5.17(iii) or by definition. Since the map

$$\mathrm{sum} : \bigoplus_{w\in S_n} \mathrm{Ext}^1_w(\pi_{\rm alg}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) \twoheadrightarrow \mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h}))$$

is surjective by [Din25, Proposition 3.8(2)], it follows that all faces are commutative. □

Since $T_\lambda^{\lambda'}$ is an exact equivalence of categories, it induces an isomorphism

$$\mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h})) \xrightarrow{\sim} \mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}'), \pi_\bullet(\phi, \mathbf{h}'))$$

and sends the universal extension associated to a subspace $W \subset \mathrm{Ext}^1_G(\pi_{\rm alg}(\phi, \mathbf{h}), \pi_\bullet(\phi, \mathbf{h}))$ to the universal extension associated to $T_\lambda^{\lambda'}(W)$. By Proposition 5.19, we have

$$T_\lambda^{\lambda'}(\pi_{\rm min}(D)) = \pi_{\rm min}(p_{\mathbf{k}}(D)) \quad \text{and} \quad T_\lambda^{\lambda'}(\pi_{\rm fs}(D)) = \pi_{\rm fs}(p_{\mathbf{k}}(D)).$$

This completes the proof of Theorem B.

Data availability. We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach. One can obtain the relevant materials from the references below.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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